Lecture III : Hypergraph product codes

Part II

In this lecture we will derive the number of encoded qubits, k, and the code distance, d, of a hypergraph product code.

Def: transpose code

Consider a linear code $\mathcal{C}$ with parity-check matrix H.
The transpose code $\mathcal{C}^T$ is the linear code with parity check matrix H^T.

Lemma 1

The number of encoded qubits, or dimension, of $\mathcal{C}^T$ is

$$\boxed{k^T = k - n + m}$$

where n is the number of physical qubits in $\mathcal{C}$ and m is the number of rows in H.

Proof

$$k^T = m - \text{rank}(H^T)$$
$$= m - \text{rank}(H) = m - (n-k)$$
$$= m - n + k$$

$\square$

Note : if H is full rank, i.e.,

$$n - m = k \quad \text{then} \quad k^T = 0 .$$

Example : repetition code

Tanner graph
$$H = \begin{bmatrix} 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 \cdots & 0 \\ 0 & \cdots & & 1\,1 \end{bmatrix}$$

$$\circ - \square - \circ - \square \cdots \square - \circ$$

Transpose code just exchanges variable and check nodes in the Tanner graph.

$$\square - \circ - \square - \circ \cdots \circ - \square$$

$$H = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & \vdots \\ \vdots & \vdots & & \\ & & & 1 \\ 0 & 0 & & 1 \end{bmatrix}$$

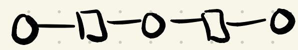
Def: subgraph product

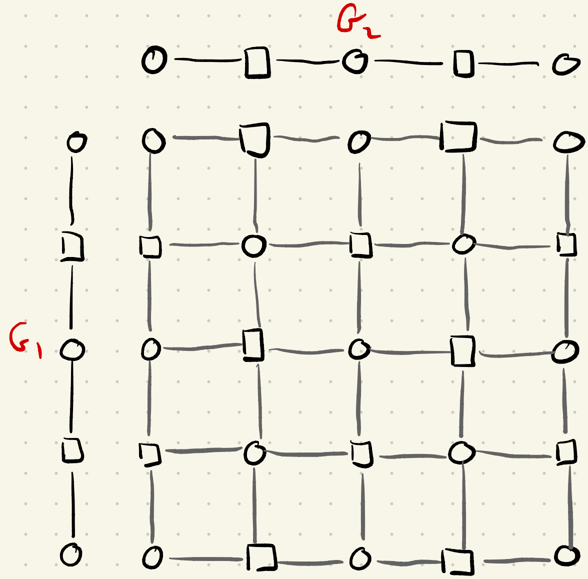
Let $G_1 = (\{V_1, C_1\}, E_1)$ and $G_2 = (\{V_2, C_2\}, E_2)$ be two Tanner graphs.

Define $G_1 \otimes G_2$ to be the induced subgraph of $G_1 \times G_2$ with variable node set $V_1 \times V_2$ and check node set $C_1 \times V_2 \cup V_1 \times C_2$.

We emphasize $G_1 \otimes G_2 \neq G_1 \times G_2$!

Example

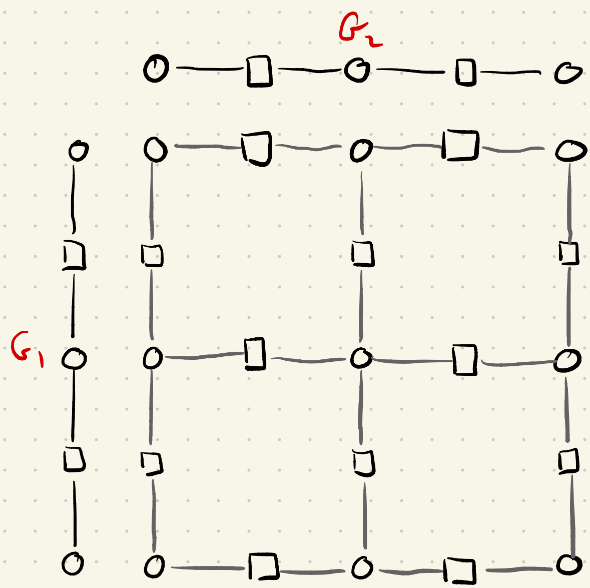
$G_1 = G_2 =$ 



$\underline{G_1 \times G_2}$

Node set

$V_1 \times V_2 \cup C_1 \times C_2$



$\underline{G_1 \otimes G_2}$

Node set

$V_1 \times V_2$

Def: product code

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two linear codes w/ parameter $[n_1, k_1, d_1]$ and $[n_2, k_2, d_2]$, respectively.

The product code $\mathcal{C}_1 \otimes \mathcal{C}_2$ is the linear code with $n = n_1 n_2$ whose codewords may be viewed as binary matrices of size $n_1 \times n_2$ such that a matrix belongs to $\mathcal{C}_1 \otimes \mathcal{C}_2$ iff all its columns belong to $\mathcal{C}_1$ and all its rows belong to $\mathcal{C}_2$.

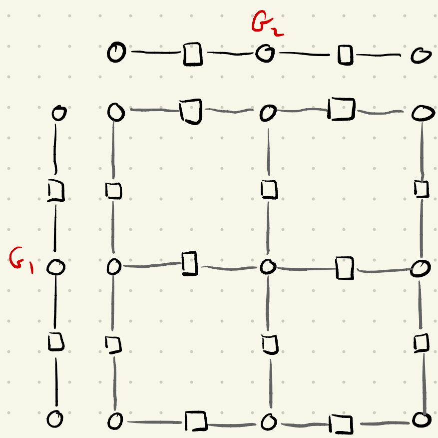
Example

$\mathcal{C}_1 = \mathcal{C}_2 = 3$ bit repetition code

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Code words $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

The corresponding Tanner graph is exactly $G_1 \otimes G_2$, where G_1 is the Tanner graph of $\mathcal{C}_1$ & G_2 is the Tanner graph of $\mathcal{C}_2$.



Lemma 2: The dimension of the product code $\mathcal{C}_1 \otimes \mathcal{C}_2$ is $k_1 k_2$ where k_i is the dimension of $\mathcal{C}_i$.

Proof

The codewords of the product code are tensor products $u^T \otimes v$ where $u \in \mathcal{C}_1$ & $v \in \mathcal{C}_2$.

We can reshape this matrix into a vector $u \otimes v$.

Recall that a generator matrix of $\mathcal{C}_i$ is a $k_i \times n_i$ matrix whose rows form a basis for $\mathcal{C}_i$.

From the form of the codewords above we observe that

$J_1 \otimes J_2$ is a generator matrix for $\mathcal{C}_1 \otimes \mathcal{C}_2$, where J_i is the generator matrix of $\mathcal{C}_i$.

$J_1 \otimes J_2$ is a $k_1 k_2 \times n_1 n_2$ matrix, hence the dimension of the product code is $k_1 k_2$. $\square$

Lemma 3: Consider the code $HGP(H_1, H_2)$, where $G_1 = (\{V_1, C_1\}, E_1)$ is the Tanner graph corresponding to H_1 and $G_2 = (\{V_2, C_2\}, E_2)$ is the Tanner graph corresponding to H_2.

Then the Tanner graph corresponding to H_X is $(G_1^T \otimes G_2)^T$ and the Tanner graph corresponding to H_Z is $(G_1 \otimes G_2^T)^T$.

Proof:

$G_1^T \otimes G_2$ has node set

$V = C_1 \times V_2$ and check set

$C = C_1 \times C_2 \cup V_1 \times V_2$

and there is an edge between

$(x, y) \in V$ & $(x', y') \in C$

if $x = x'$ and $\{y, y'\} \in E_2$

or $y = y'$ and $\{x, x'\} \in E_1$.

$(G_1^T \otimes G_2)^T$ has vertex set

$C = C_1 \times V_2$

$V = V_1 \times V_2 \cup C_1 \times C_2$ and the same edge set.

This is exactly the subgraph of $G_1 \times G_2$ induced by $C_1 \times V_2$ ie the

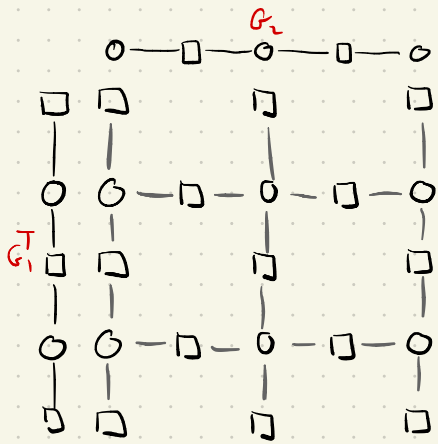
the Tanner graph of H_X.

The argument for H_Z is analogous. $\square$

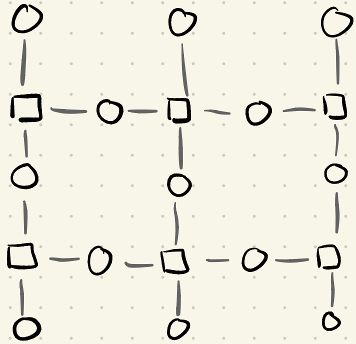
<u>Example</u> $H_1 = H_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$G_1 = G_2 =$ 

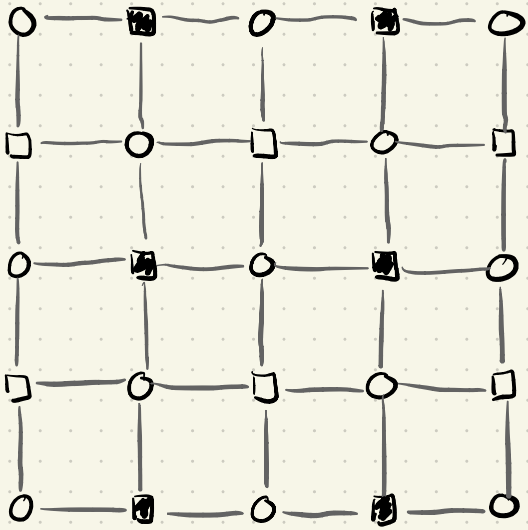
$G_1^T \otimes G_2$



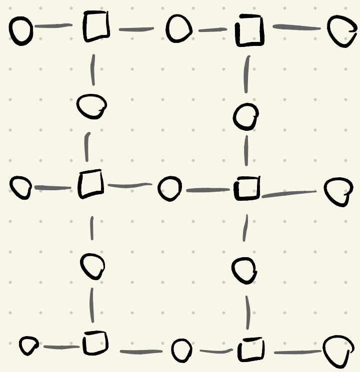
$\left(G_1^T \otimes G_2\right)^T$



$HGP(H_1, H_2)$



$\left(G_1 \otimes G_2^T\right)^T$



Lemma 4: The number of encoded qubits of the code $HGP(H_1, H_2)$ is $\boxed{k = k_1 k_2 + k_1^T k_2^T}$, where $H_1 \in \mathcal{M}_{m_1 \times n_1}(\mathbb{F}_2)$ is the pcm of linear code $\mathcal{C}_1$ w/ parameters $[n_1, k_1, d_1]$ and $H_2 \in \mathcal{M}_{m_2 \times n_2}(\mathbb{F}_2)$ is the pcm of linear code $\mathcal{C}_2$ w/ parameters $[n_2, k_2, d_2]$.

Proof:

$$k = n - \mathrm{rank}(H_X) - \mathrm{rank}(H_Z)$$

$$= n - (n - \dim(\mathcal{C}_X)) - (n - \dim(\mathcal{C}_Z))$$

linear code defined by H_X

$$k = \dim(\mathcal{C}_X) + \dim(\mathcal{C}_Z) - n$$

$$\dim(\mathcal{C}_X) = n_1 n_2 + m_1 m_2 - m_1 n_2 + \dim(\mathcal{C}_X^T)$$ Lem. 1

$$= n_1 n_2 + m_1 m_2 - m_1 n_2 + \dim(\mathcal{C}_1^T \otimes \mathcal{C}_2)$$ Lem. 3

$$= n_1 n_2 + m_1 m_2 - m_1 n_2 + k_1^T k_2$$ Lem. 2

$$\dim(\mathcal{C}_Z) = n_1 n_2 + m_1 m_2 - n_1 m_2 + \dim(\mathcal{C}_Z^T)$$ Lem. 1

$$= n_1 n_2 + m_1 m_2 - n_1 m_2 + k_1 k_2^T$$

$$k = n_1 n_2 + m_1 m_2 - m_1 n_2 + k_1^T k_2 + \cancel{n_1 n_2} + m_1 m_2 - n_1 m_2 + k_1 k_2^T - \cancel{n_1 n_2}$$

$$k = n_1 n_2 + m_1 m_2 - m_1 n_2 + k_1^T k_2$$
$$+ m_1 m_2 - n_1 m_2 + k_1 k_2^T$$
$$= n_1 (n_2 - m_2) - m_1 (n_2 - m_2)$$
$$+ m_1 m_2 \quad + k_1^T k_2 + k_1 k_2^T$$
$$= (n_1 - m_1)(n_2 - m_2)$$
$$+ k_1^T k_2 + k_1 k_2^T$$
$$= (k_1 - k_1^T)(k_2 - k_2^T) \qquad \text{Lemma 1}$$
$$+ k_1^T k_2 + k_1 k_2^T$$
$$= k_1 k_2 - \cancel{k_1 k_2^T} - \cancel{k_1^T k_2} + k_1^T k_2^T$$
$$+ \cancel{k_1^T k_2} + \cancel{k_1 k_2^T}$$
$$= k_1 k_2 + k_1^T k_2^T$$

$\square$

Lemma 5: The code distance

of the code $HGP(H_1, H_2)$

$\boxed{d \geqslant \min(d_1, d_2, d_1^T, d_2^T)}$, where

$H_1 \in \mathcal{M}_{m_1 \times n_1}(\mathbb{F}_2)$ is the pcm of linear code

$\mathcal{C}_1$ w/ parameters $[n_1, k_1, d_1]$ and

$H_2 \in \mathcal{M}_{m_2 \times n_2}(\mathbb{F}_2)$ is the pcm of linear code

$\mathcal{C}_2$ w/ parameters $[n_2, k_2, d_2]$.

Proof

Consider a Pauli Z-type operator

that commutes with the X-type

stabilizes of $HGP(H_1, H_2)$ and

has weight $\leqslant \min(d_1, d_2^T)$.

Such an operator can be represented by a codeword $z \in \mathcal{C}_X$, where $\mathcal{C}_X$ is the linear code defined by H_X.

Let $\mathrm{supp}(z) \subseteq V_1 \times V_2 \cup C_1 \times C_2$ denote the support of z.

Define

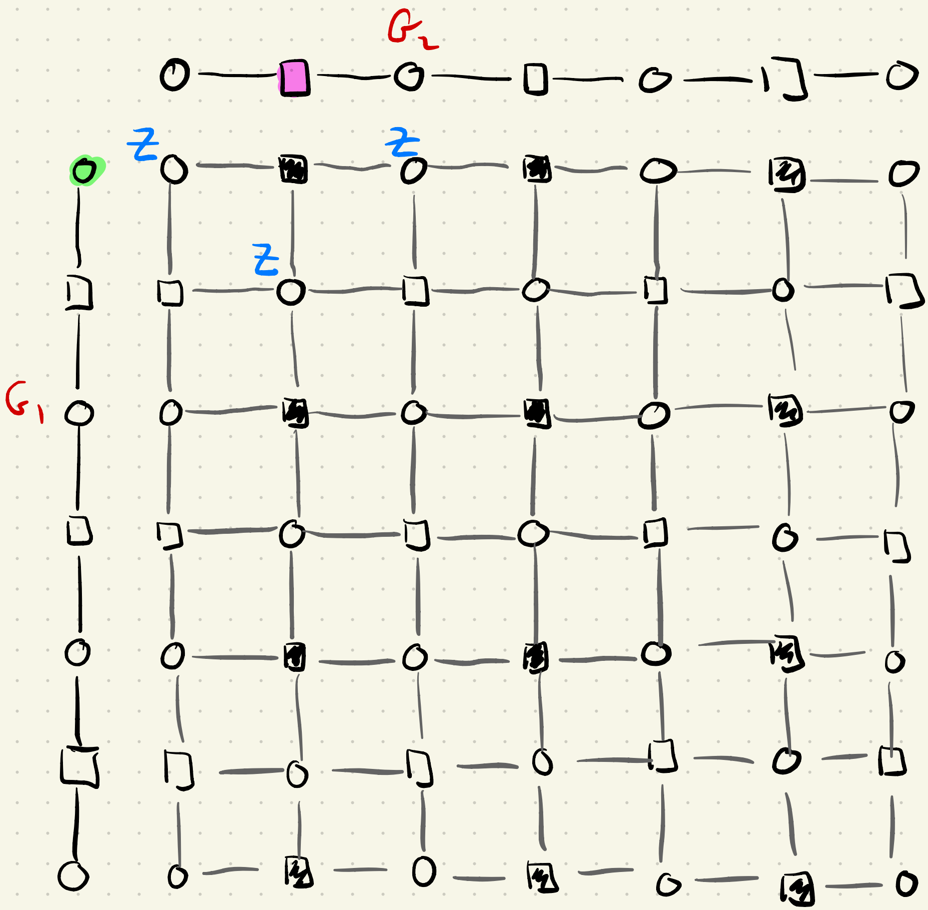
$$V_1' := \{ v' \in V_1 : \exists v \in V_2, (v', v) \in \mathrm{supp}(z) \}$$

$$C_2' := \{ c' \in C_2 : \exists c \in C_1, (c', c) \in \mathrm{supp}(z) \}$$

Let G_1' be the subgraph of G_1 (Tanner graph of H_1) induced by $V_1' \cup C_2$ and let G_2' be the subgraph of G_2 induced by $V_1 \cup C_2'$.

Example :

$$H_1 = H_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

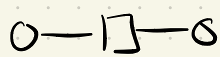


z

V_1'

C_2'

G_1' 

G_2' 

Let $\mathcal{C}_i$ and $\mathcal{C}_i'$ be the linear code defined by the Tanner graphs G_i & G_i', respectively.

Any code word of $\mathcal{C}_1'$ can be extended to a codeward of $\mathcal{C}_1$ by padding it with zeros.

$(x_0, x_1, \dots, x_{|V_1'|}) \in \mathcal{C}_1'$

$(x_0, x_1, \dots, x_{|V_1'|}, 0, 0, \dots, 0) \in \mathcal{C}_1$

We also have $|V_1'| < d_1$ and

so $x_0 = x_1 = \dots = x_{|V_1'|} = 0$, i.e.

$\dim(\mathcal{C}_1') = 0$.

Similarly any codeword of $\mathcal{C}_2'^T$ can be extended to a codeword of $\mathcal{C}_2^T$ by padding w/ zeros but the codeword has wt $\leq d_2^T$ so $\dim(\mathcal{C}_2'^T) = 0$.

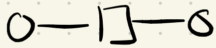
Let H_i' be the pcm corresponding to G_i'.

By Lemma 4, the code HGP(H_1', H_2') has

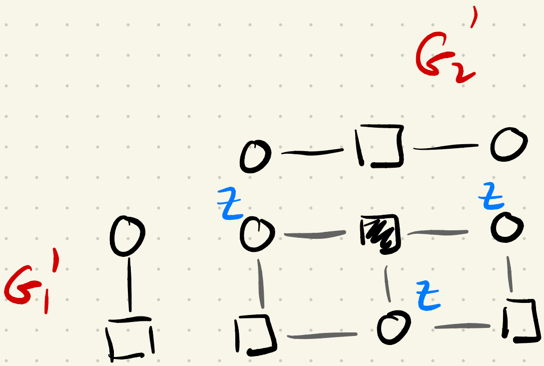
$$k' = \underset{\substack{\uparrow \\ 0 \\ \dim(\mathcal{C}_1')}}{k_1'}\, k_2' + k_1'^T \underset{\substack{\uparrow \\ 0 \\ \dim(\mathcal{C}_2'^T)}}{k_2'^T} = 0$$

Denote by $\mathfrak{z}'$ the restriction of $\mathfrak{z}$ to $V_1' \times V_2 \cup C_1 \times C_2'$.

Example

G_1'  G_2' 

$HGP(H_1', H_2')$



$\mathfrak{z}'$

Denote by $h_z(v_1, c_2)$ the row of H_z corresponding to $v_1 \in V_1$ & $c_2 \in C_2$ and similar for $h_z'(v_1', c_2')$ with $v_1' \in V_1'$ and $c_2' \in C_2'$.

Note that as $k' = 0$ any operator $\mathfrak{z}'$ that commutes with the X-type stabilizers of $HGP(H_1', H_2')$ must be a Z-type stabilizer of $HGP(H_1', H_2')$.

$$\mathfrak{z}' = \bigoplus_{(v_1', c_2') \in J} h_z'(v_1', c_2') \quad \text{where}$$

$$J \subseteq V_1' \times C_2'$$

The set of neighbours of any $(v_1', c_2') \in V'$ is the <u>same</u> as in the corresponding $(v_1', c_2') \in V$.

<u>Recall</u> the vertex set of G_1' is $V_1' \cup C_1$ and the vertex set of G_2' is $V_2 \cup C_2'$, and $v_1' \in V_1'$, $c_2' \in C_2'$.

$\Rightarrow$ The vertex set of $G_1' \times G_2'$ is $V_1' \times V_2 \cup C_1 \times C_2'$.

All the neighbours of (v_1', c_2') in $G_1 \times G_2$ are contained in $\{v_1'\} \times V_2 \cup C_1 \times \{c_2'\}$, which is a subset of the above.

Therefore we also have

$$\mathfrak{z} = \bigoplus_{(v_1', c_2') \in J} h_Z(v_1', c_2') \quad \text{ie}$$

$\mathfrak{z}$ is a Z-type stabilizer.

Therefore any Z-type operator that commutes w/ the X stabilizers w/ weight $\leq \min(d_1, d_2^T)$ must be a Z-type stabilizer.

Running the same argument w/ X & Z exchanged allows us to conclude that any X-type operator that commutes w/ the

Z stabilizers and has weight $\leq \min(d_1^T, d_2)$ must be an X-type stabilizer.

Therefore any operator that commutes with all the stabilizers and is not itself a stabilizer has weight $\geq \min(d_1, d_2, d_1^T, d_2^T)$. $\square$

One can also show that the code distance of $HGP(H_1, H_2)$

$$d \leq \min(d_1, d_2, d_1^T, d_2^T)$$

and therefore

$$\boxed{d = \min(d_1, d_2, d_1^T, d_2^T)}$$

We can therefore conclude that the hypergraph product code $HGP(H_1, H_2)$ has parameters

$$\boxed{[[n_1 n_2 + m_1 m_2 ,\ k_1 k_2 + k_1^T k_2^T ,\ \min(d_1 , d_2 , d_1^T , d_2^T)]]}$$