Lecture VI

The threshold theorem:

Proof and assumptions

Recall: an error model

is <u>local stochastic</u> if

for any set of faults

R, $\Pr[S \subseteq R]$

$$= \sum_{R \mid S \subseteq R} \Pr[R] \leq p^{|S|}$$

where $0 < p < 1$.

③

$$\Pr[S \subseteq R] = \sum_{R \mid S \subseteq R} \Pr[R]$$

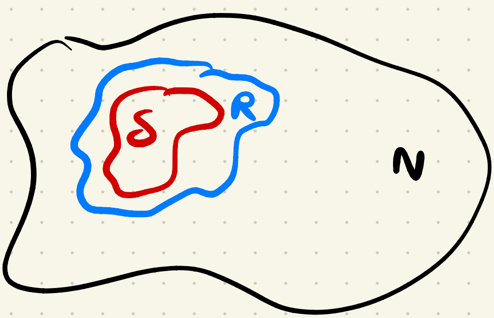
iid $\quad \Pr[R] = p^{|R|}(1-p)^{N-|R|}$

N total # of locations

$$\sum_{R \mid S \subseteq R} \Pr[R] =$$

$$p^{|S|} \sum_{R \mid S \subseteq R} p^{|R|-|S|}(1-p)^{(N-|S|)-(|R|-|S|)}$$

$$= p^{|S|} \underbrace{\sum_{T} p^{|T|}(1-p)^{M-|T|}}_{=1}$$



$M = N - |S|$

T is any subset of the M locations

Recall: level reduction

Given a circuit C we can construct a FT circuit, which when subjected to local stochastic noise w/ error rate p, is equivalent to C subjected to local stochastic noise w/ error rate $p' = \binom{A}{t+1} p^{t+1}$

locations in largest exRec

If $\binom{A}{t+1} p^{t+1} < p$ then our FT circuit is more reliable than C.

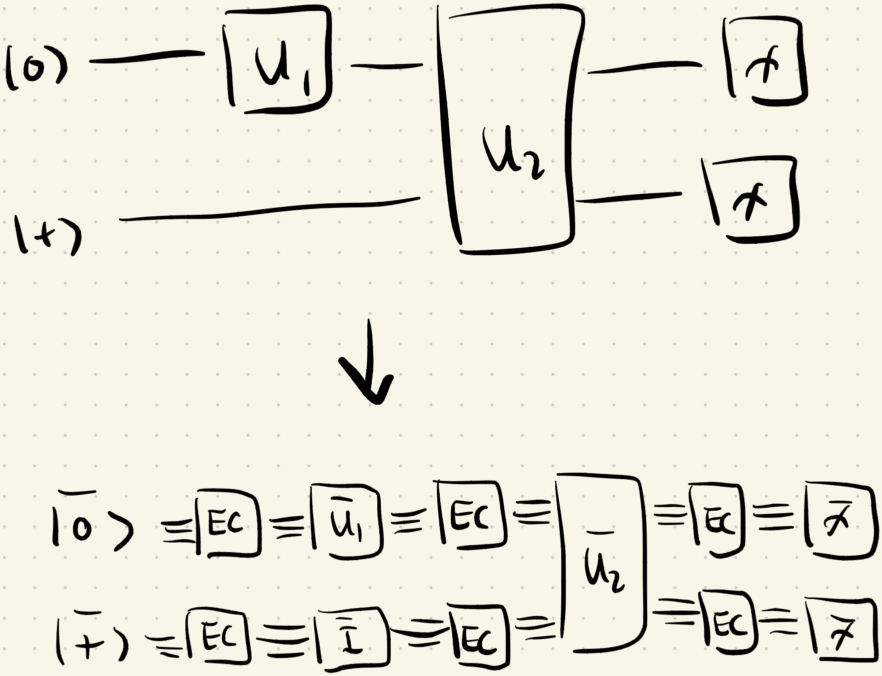
We can repeat this process to further reduce the error rate.

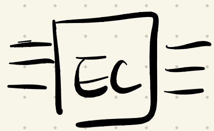
Def: Code concatenation

Given an $[[n,1,d]]$ code we take each physical qubit of the code and encode it again using the same code, giving an $[[n^2, 1, d^2]]$ code.

Repeating this L times gives a $[[n^L, 1, d^L]]$ code.

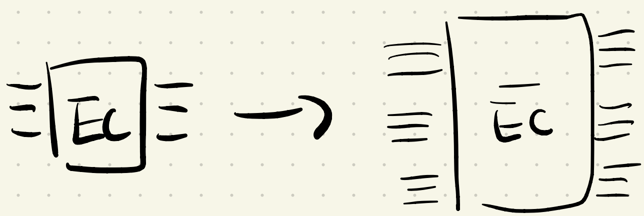
In a similar way we can define a concatenated FT simulation of a circuit.





is a circuit

We encode this again in the same code



Thm: $\exists$ p_T such that

if a system is subjected to local stochastic noise w/ error prob. $p < p_T$, then for any $\varepsilon > 0$ & any circuit C with T locations, there exists a FT circuit with output distribution within statistical distance ε of the output distribution of C (executed perfectly).

The FT protocol uses resources (time, qubits, gates) that are a factor polylog(T/ε) greater than those of C.

Proof: Idea is to use a concatenated FT sim. w/ L levels.

1st level of concatenation

$$p^{(1)} \leq \binom{A}{t+1} p^{t+1} \qquad t = \left\lfloor \frac{d-1}{2} \right\rfloor$$

Define $p_T = 1 \Big/ \binom{A}{t+1}^{1/t}$

$$p^{(1)} \leq p_T \left(\frac{p}{p_T} \right)^{t+1}$$

$$\frac{p^{(1)}}{p_T} = \left(\frac{p}{p_T} \right)^{t+1}$$

2nd level of concatenation

$$\frac{p^{(2)}}{p_T} \leq \left(\frac{p^{(1)}}{p_T} \right)^{t+1}$$

$$= \left(\left(\frac{p}{p_T} \right)^{t+1} \frac{p_T}{p_T} \right)^{t+1}$$

$$= \left(\frac{p}{p_T} \right)^{(t+1)^2}$$

L'th level of concatenation

$$\frac{p^{(L)}}{p_T} \leq \left(\frac{p}{p_T}\right)^{(t+1)^L}$$

If $p < p_T$ we can make the error arbitrarily small by choosing L large enough

We choose

$$L = \left\lceil \log_{t+1} \log_{p/p_T} (\varepsilon / p_T T) \right\rceil$$

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$$= \left\lceil \frac{\log_2 \left(\frac{\log_2 (T p_T / \varepsilon)}{\log (p_T / p)} \right)}{\log_2 (t+1)} \right\rceil$$

$$= O(\log \log (T/\varepsilon))$$

This gives

$$p^{(L)} \leq \varepsilon / T$$

$\Rightarrow$ Prob of having a single logical fault is $\leq \varepsilon$ (T locations in circuit) $\square$

This gives a lower bound on the error threshold p_T, but what is p_T in practice? (Billion dollar question!)

<u>Example</u>: Concatenated Steane code

$d=3 \quad t=7$

$p_T = 1/\binom{A}{2}$

One can calulate e.g.

$A = 679$

$\Rightarrow \binom{A}{2} = 230{,}181$

$p_T = 4 \cdot 3 \times 10^{-6}$

Highest proven threshold value is for Knill's scheme where $p_T > 10^{-3}$

In practice people often estimate the threshold using simulations.
(possible due to Gottesman Knill theorem)

For Knill's scheme

$p_T \sim 3\%$

For surface code

$p_T \sim 1\%$

In practice the polylog overhead can hide large constant factors.

e.g. surface code

$\sim 10^3$ physical qubits needed per logical qubit!

But using certain special codes (low-density parity-check codes w/ additional properties) one can show that

FT q. comp. is possible w/ constant overhead!

Reducing the overhead for practical FT schemes is a v. important research problem!

Assumptions behind the threshold thm

1. Same error rates for all locations

Not necessary

We can repeat our proof but now p_T not a number but a surface.

2 Local error model

Necessary

Small-scale correlation
is included in local stochastic
error model

But long range correlation
will kill the threshold

This is a real problem

e.g. cosmic rays in
superconducting circuits

3 Long range gates

Not necessary

In concatenated codes we need we naively need long range connections between qubits.

We can avoid this by using SWAP gates or topological codes

4 Stochastic errors

Not necessary (not fully proven)

There exists a threshold
than for coherent errors
but with a reduced
threshold (sim. for
non-Markovian errors).
But it's not clear if
this is a real effect
or an artefact of the

proof technique.

Coherent errors &

non-Markovian errors

are difficult to simulate,

so we don't have much

numerical evidence one way

or another.

References

- Daniel Gottesman 2020 notes for IQC QEC class
- Dan E. Browne 2017 QEC course notes
- C. Chamberland & M. Beverland, FT EC w/ arb. dist. codes

https://doi.org/10.22331/q-2018-02-08-53

- Nielsen & Chuang
  Quantum Computation and
  Quantum Information

- Eastin & Knill
  Restrictions on Transversal Encoded
  Q. Gate Sets arXiv:0811.4262

- Dawson & Nielsen

  The Solovay-Kitaev alg.

  https://arxiv.org/abs/quant-ph/0505030

- Nebe, Rains, Sloane
  The invariants of the Clifford

  groups https://arxiv.org/abs/math/0001038

- Bravyi, Kitaev

  Universal q. comp. w/ ideal Clifford gates and noisy ancillas

  https://arxiv.org/abs/quant-ph/0403025

- Daniel Gottesman

  Surviving as a Quantum Computer in a Classical World

- E. Knill, Q. comp w/ realistically noisy devices

  https://doi.org/10.1038/nature03350

- D. Gottesman

  FT Q. Comp. w/ Constant Overhead

  https://arxiv.org/abs/1310.2984