Lecture $\overline{\underline{V}}$

The Threshold Theorem:

Definitions & level reduction

In the next two lectures we will prove the threshold theorem, one of the most important results in quantum information theory.

Fault-tolerant EC (formal def)

Def: FT EC Recovery Property (ECRP) 2 in Lecture 1 p. 13

$\boxed{EC}$ satisfies ECRP if whenever $s \leq t$ $\quad t = \lfloor \frac{d-1}{2} \rfloor$

$$\equiv \boxed{EC} \equiv \;=\; \equiv \boxed{EC} \equiv \boxed{f_s} \equiv$$

$\uparrow$ s faults

where $\boxed{f_s}$ is an s-filter

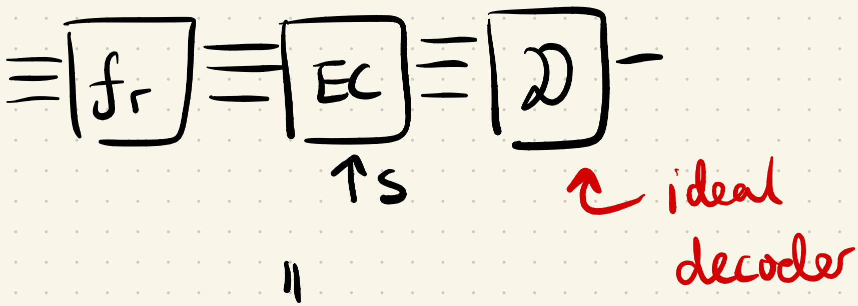
ie a projector onto the subspace spanned by codewords w/ $\leq s$ errors

Def: FT EC Correctness Property (ECCP)

1 in lecture 1 p.12

$\boxed{EC}$ satisfies ECCP if

whenever $r + s \leq t$



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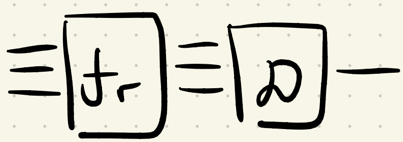
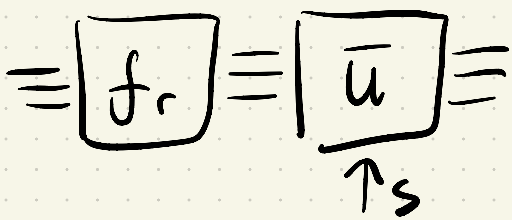


Def: FT Gate Error Propagation Property (GPP)

1q logical gate $\boxed{\bar{u}}$ satisfies GPP if, whenever $r+s \le t$,



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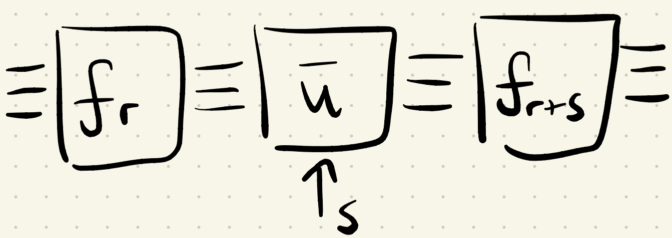
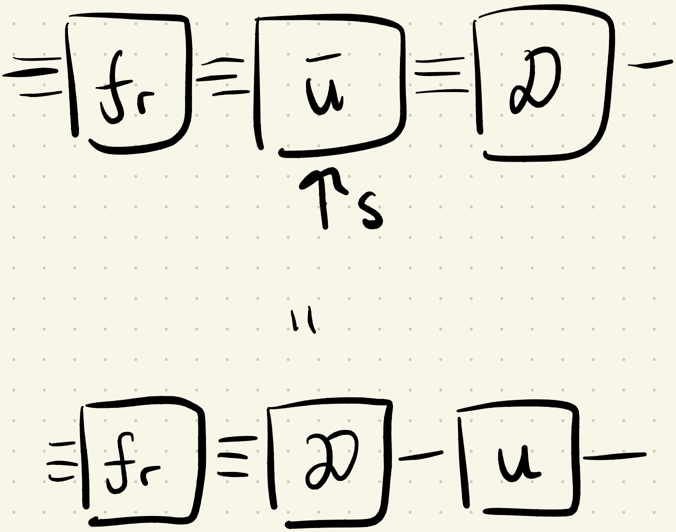


Def : FT Gate Correctness Property (GCP)

A 1q logical gate $\boxed{\bar{u}}$ satisfies GCP if, whenever $r+s \leq t$



ideal gate

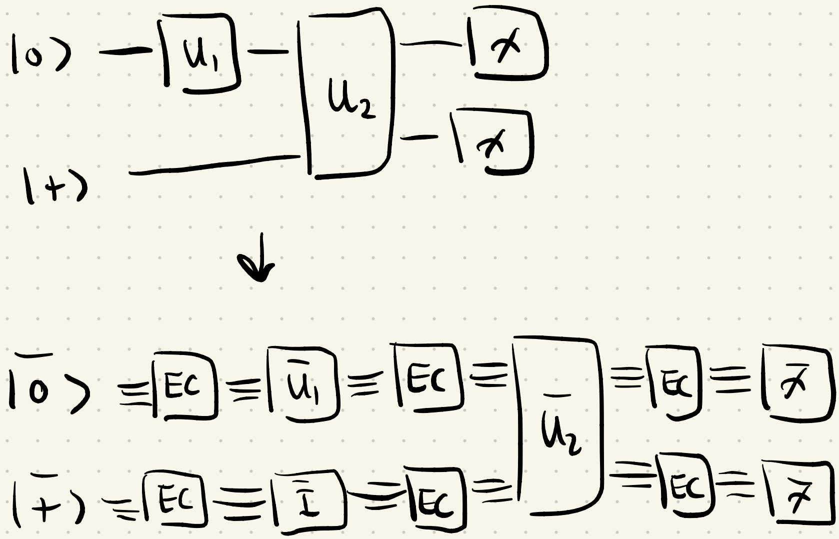
Can check these conditions for the FT EC & gate implementations we covered in prev. lectures.

(Also analogous defs for state prep. & meas.)

We assume ECRP, ECCP, GPP, GCP hold.

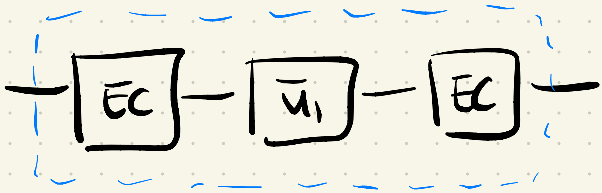
Extended rectangles (exRecs)

Recall the fault-tolerant version of a circuit

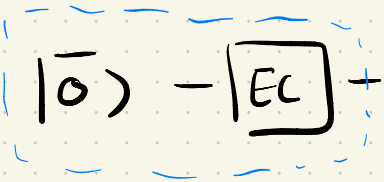


(Encoding in $k=1$ QECC)

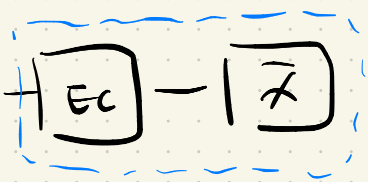
An exRec consists of



Logical gate + EC before & after



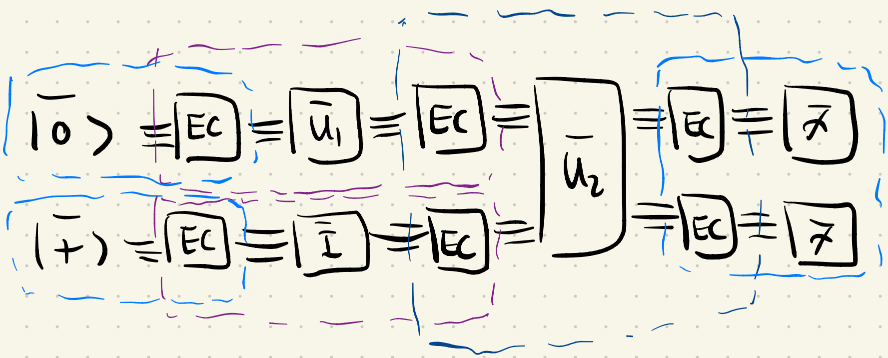
Logical state prep + EC after



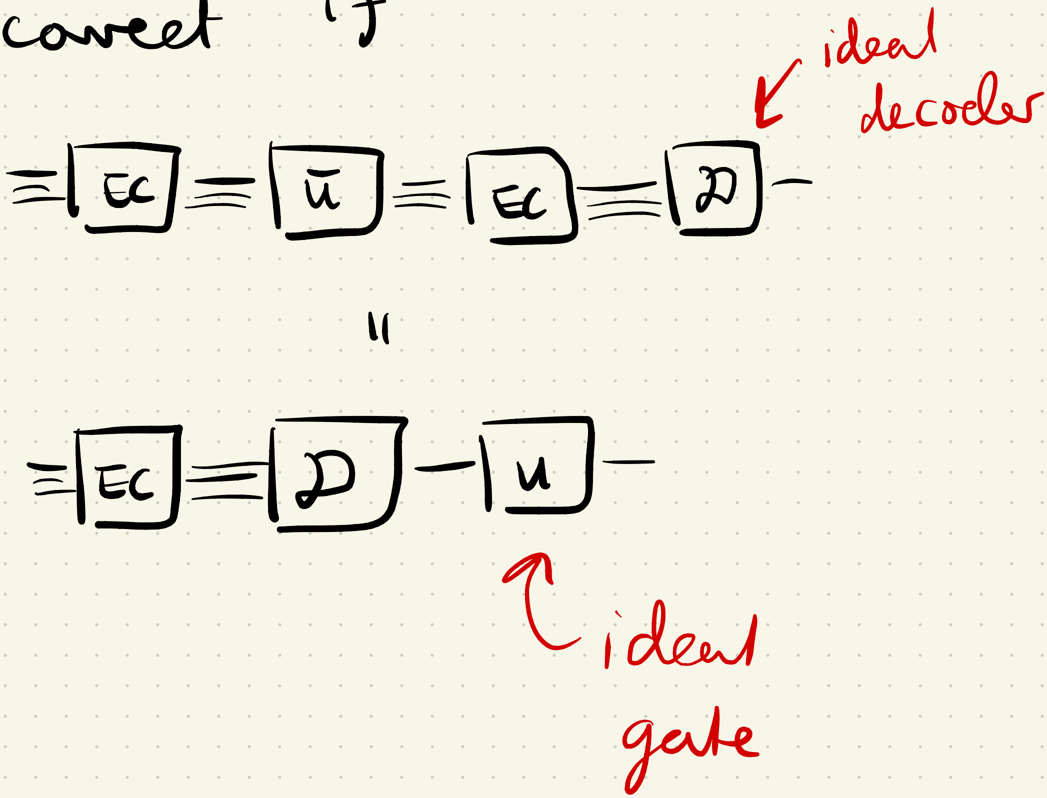
Logical measurement + EC before

If the underlying QECC corrects t errors, an exRec is good if it contains no more than t faults and is bad otherwise.

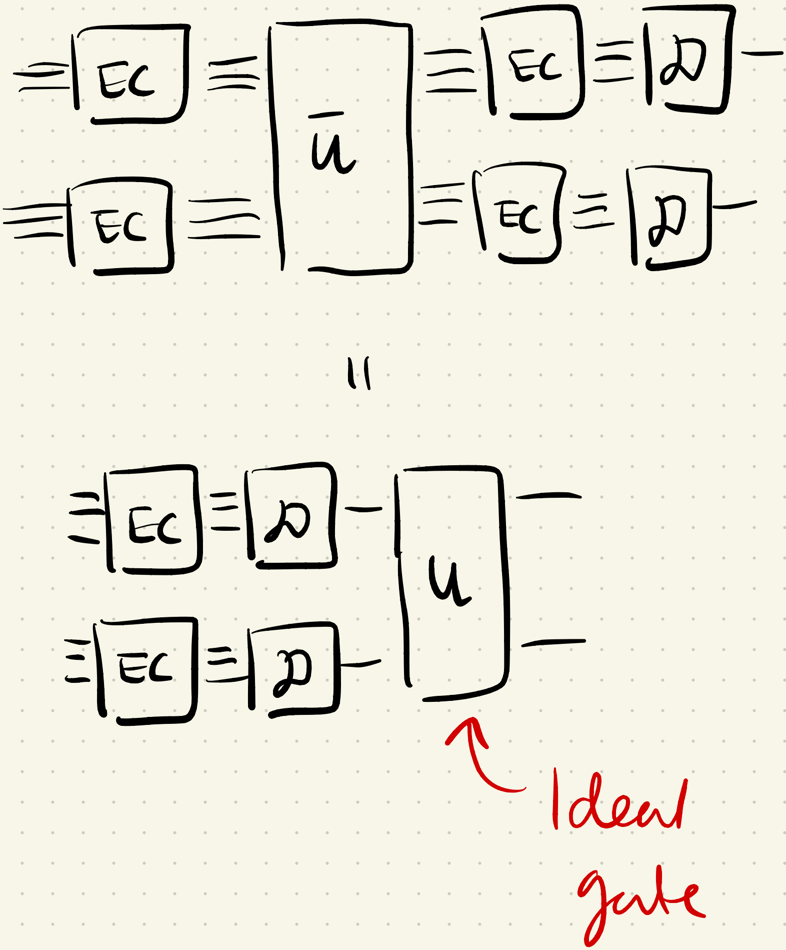
exRecs overlap



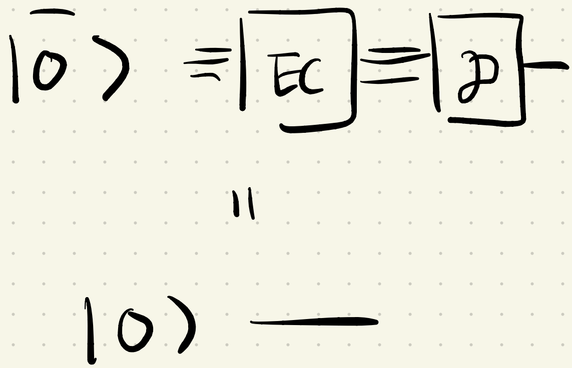
A 1q gate exRec is correct if



A 2q gate exRec is correct if

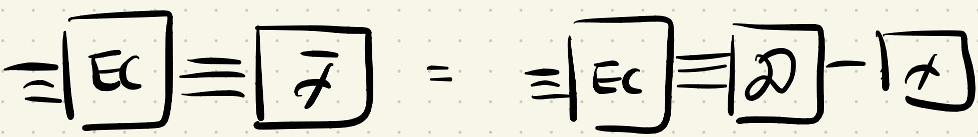


A state prep. exRec is correct if



ideal state prep.

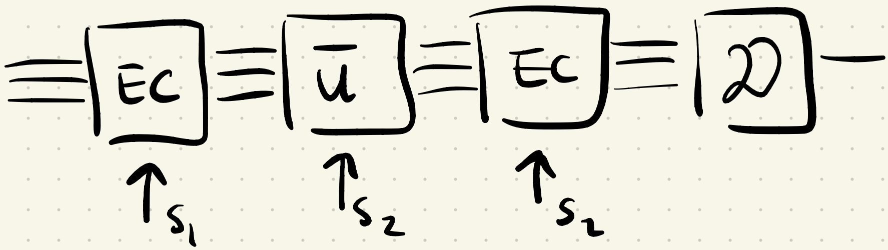
A measurement exRec is correct if



ideal measurement

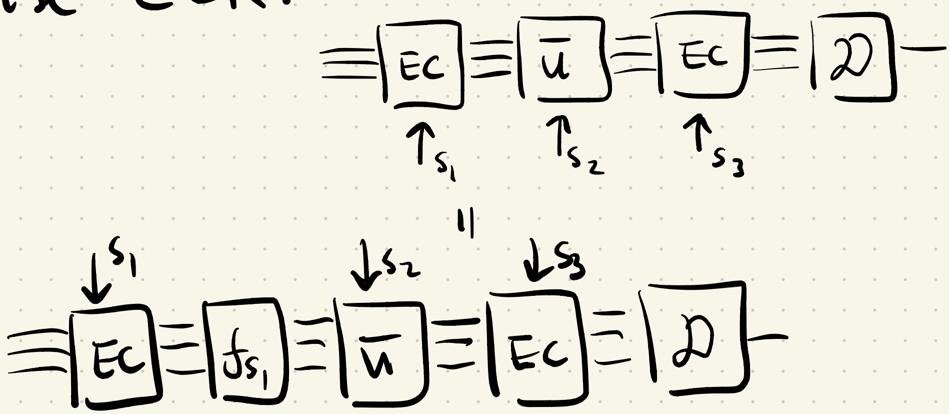
Thm: A good exRec is a correct exRec

Proof (1q gates)

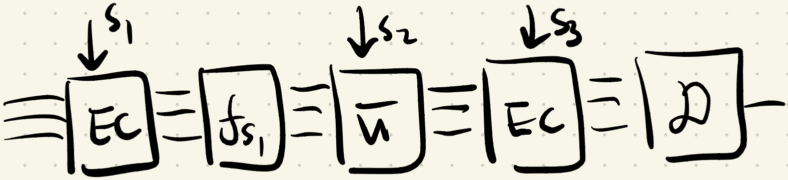


$s_1+s_2+s_3 \le t$ (Good exRec)

Use ECRP



Use GPP



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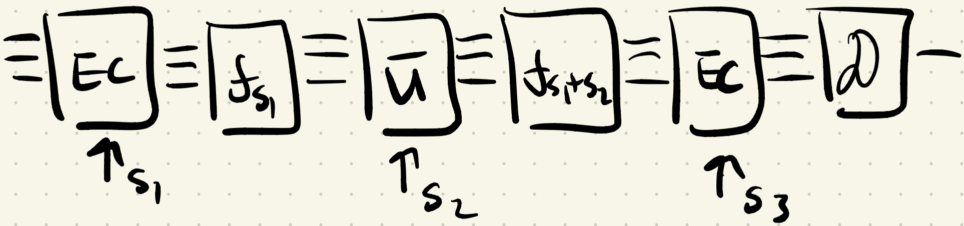
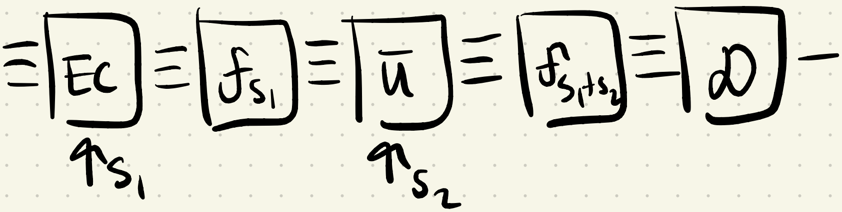
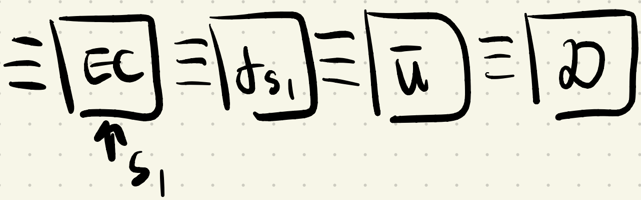


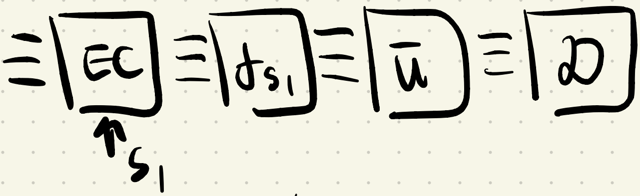
= (ECCP)



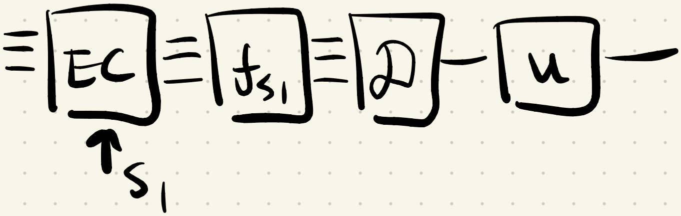
= (reverse GPP)



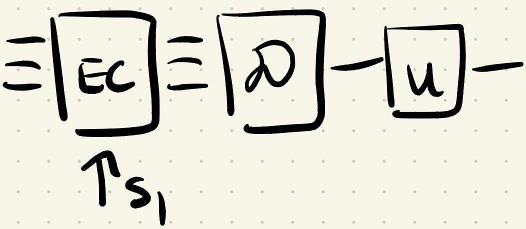
Use GCP



$=$



$=$ (reverse ECRP)

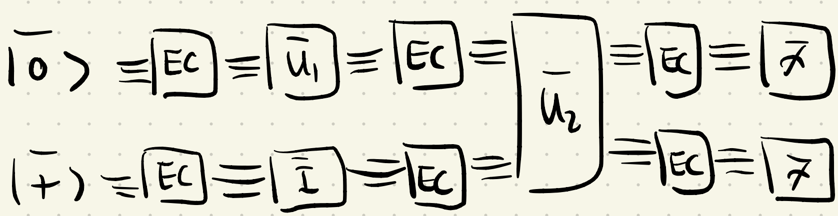


$\square$

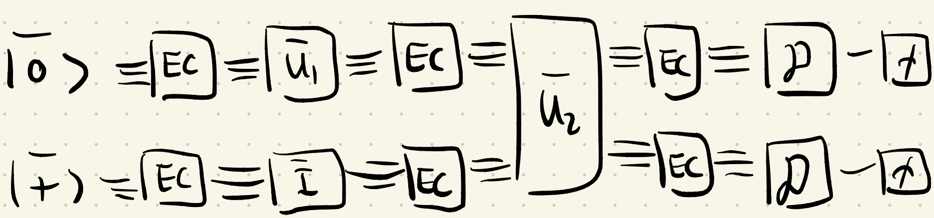
Proof for other location types is similar

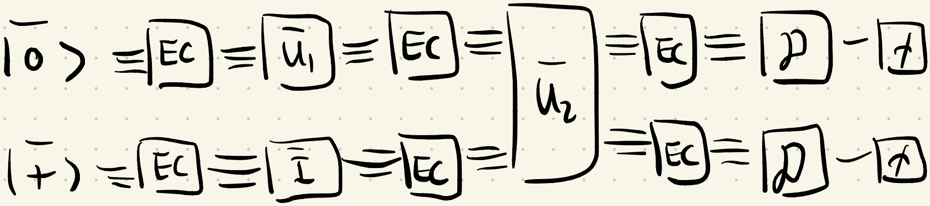
Corr: If all exRecs in an FT circuit are good then the output distribution is the same as the output distribution of the ideal circuit.

Proof sketch:

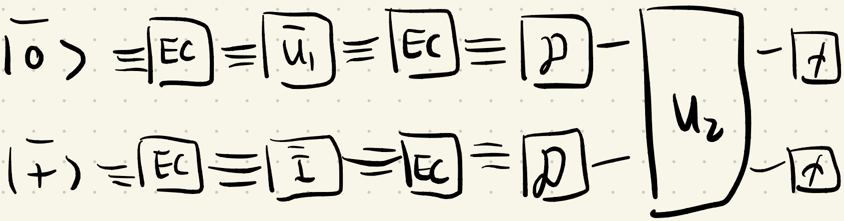


= (Correct meas. exRecs)

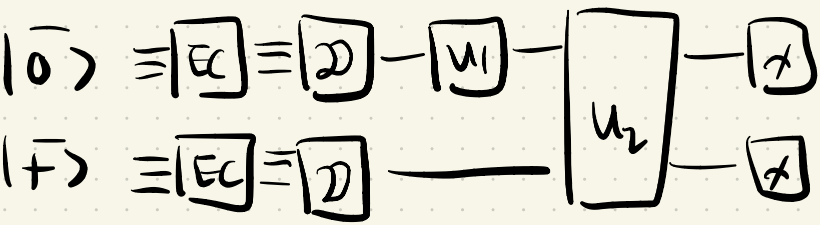




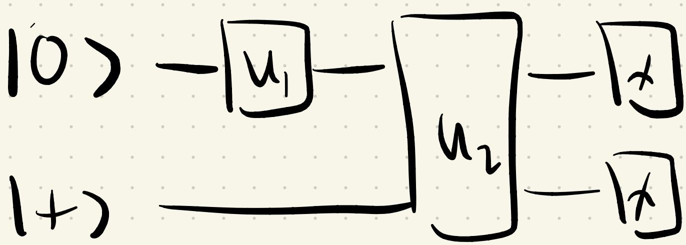
= (Correct 2q exRec)



= (Correct 1q exRecs)



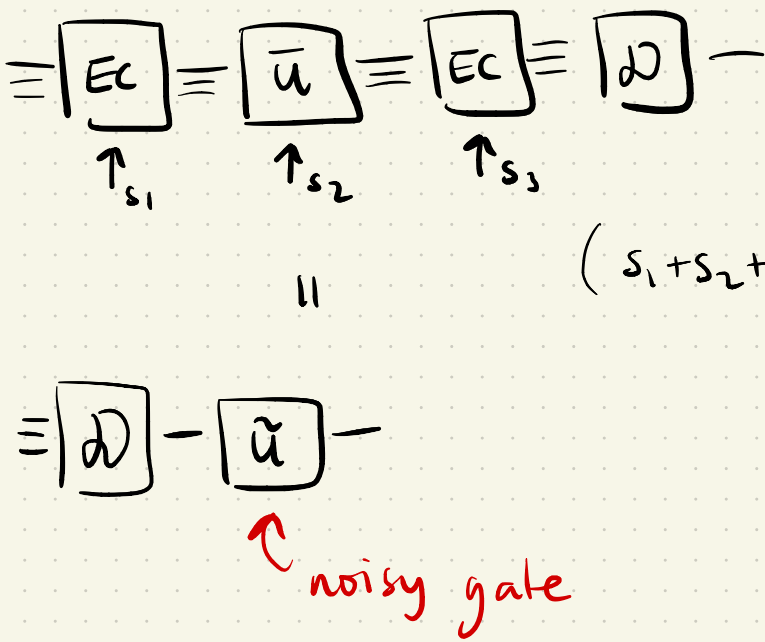
= (Correct state prep. exRecs)



□

But what if an exRec is bad?

We want to replace bad exRecs with faulty locations e.g.



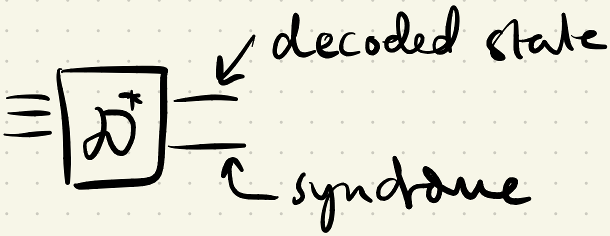
$(s_1 + s_2 + s_3 > t)$

But the noisy gate $\tilde{U}$ will in general depend on the noise in the exRec *and* the error syndrome of the state entering the exRec.

The solution is to keep track of the error syndrome information.

Def : $*$ - decoder $\boxed{\mathcal{D}^*}$

works just like the ideal decoder but keeps the syndrome information



One can show

$$=\boxed{\mathrm{Ec}}=\boxed{\bar{U}}=\boxed{\mathrm{Ec}}=\boxed{\mathcal{D}^*}-$$

$$=$$

$$=\boxed{\mathcal{D}^*}-\boxed{\tilde{U}}-$$

noisy gate that depends on syndrome

$*$ - decoder is unitary U_*

Let $E = \;=\boxed{\mathrm{Ec}}=\boxed{\bar{U}}=\boxed{\mathrm{Ec}}=$

$$EU_* = U_* \underbrace{U_*^\dagger E U_*}_{\text{noisy gate } \tilde{U}}$$

Recap

We have shown that we can replace good exRecs by ideal locations and bad exRecs by faulty locations.

The obvious next question is: how many bad exRecs do we expect?

To answer this question we must first choose a noise model.

Def: an error model is local stochastic if for any set of faults R, $\Pr[S \subseteq R]$

$$= \sum_{R \mid S \subseteq R} \Pr[R] \leq p^{|S|}$$

where $0 < p < 1$.

This is more general than iid noise and can include adversarial noise.

Assuming a local stochastic noise model, what is the prob. that a single exRec is bad?

Suppose the exRec contains A locations.

exRec is bad if it contains $>t$ faults.

Then

union bound

$$\Pr[\text{exRec bad}] \leq \sum_{|S|=t+1} \sum_{R \mid S \subseteq R} \Pr[R]$$

includes sets R of size $t+2$ etc. (In fact we overcount these.)

$$= \binom{A}{t+1} p^{t+1}$$

We can now prove our main theorem, the level reduction theorem.

Thm : Suppose we have a fault tolerant circuit simulating an unencoded circuit C, subjected to a local stochastic noise model w/ error prob. p.

Then the FT circuit is equivalent to C subjected to a local stochastic noise model w/ error prob. p' where

$$p' \leq \binom{A}{t+1} p^{t+1}$$

and A is the number of locations in the largest exRec in the FT circuit.

Proof :

For any given run of the FT circuit there will be some set of faults R that occur w/ prob. $Pr[R]$

For R we assign good and bad exRecs.

Then the FT circuit is equivalent to C w/ faults at the locations corresponding to bad exRecs.

We need to show:

given a set of r exRecs in the FT circuit the probability that there exists a set of faults R such that each of the r exRecs is bad is at most p'^r.

The set of exRecs is bad if every exRec has $t+1$ or more faults.

There are at most

$\binom{A}{t+1}^r$ sets of locations

with $t+1$ locations in each

of the r exRecs.

Each such set has a

total of $(t+1)r$ locations.

$\Rightarrow$ Prob. of a set of

faults containing all these

locations is $\leq p^{(t+1)r}$

Union bound

$\Rightarrow$ Prob. of $t+1$ faults on each of the r exRecs is

$$\binom{A}{t+1}^r p^{(t+1)r} = \left[\binom{A}{t+1} p^{t+1}\right]^r$$

ie local stochastic noise w/ error prob. $p' = \binom{A}{t+1} p^{t+1}$

$\square$.

[There is a subtlety about overlapping exRecs that I have neglected, see Gottesman notes]