Lecture 5: Entanglement

Consider the state

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

$$\mathrm{Tr}_A\left(|\phi^+\rangle\langle\phi^+|\right)$$

$$= \mathrm{Tr}_A\left(\frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)\right)$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$= \frac{1}{2}\hat{I}$$

Access to A subsystem gives us <u>no information</u> about the state!

- Initially considered a mysterious aspect of q. mechanics.

  Einstein, Podolsky, Rosen (1935)

- Bell gave a quantitative analysis : Bell inequalities (1964)

  Metaphysical implications.

- Experimental validation by Aspect at Institut d'optique theorique et appliquée (Orsay)!

  Nobel prize in 2022 for this work (shared with Clauser e Zeilinger).

Bipartite pure states

A state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is entangled if it cannot be written as a product state

$$|\psi\rangle_{AB} \neq |\varphi\rangle_A \otimes |\chi_B\rangle.$$

Equivalently, $|\psi\rangle_{AB}$ is entangled if its Schmidt number is > 1.

Recall Schmidt decomp:

$$|\psi\rangle_{AB} = \sum_{i=1}^{d} \lambda_i |\varphi_i\rangle_A \otimes |\chi_i\rangle_B$$

orthonormal

Schmidt # d

Bell states

$$|\phi^{\pm}\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} \pm |11\rangle_{BB}\right)$$

$$|\psi^{\pm}\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|01\rangle_{AB} \pm |10\rangle_{AB}\right)$$

$$|\phi^{-}\rangle = \hat{Z}\otimes\hat{I}\,|\phi^{+}\rangle$$

$$|\psi^{-}\rangle = \hat{Z}\otimes\hat{I}\,|\psi^{+}\rangle$$

$$|\psi^{+}\rangle = \hat{X}\otimes\hat{I}\,|\phi^{+}\rangle$$

Basis for 2-qubit states

Local operations i.e. $\mathcal{E} = \mathcal{E}_A \otimes \mathcal{E}_B$

cannot change entanglement

structure.

$$\mathcal{E}_A \otimes \mathcal{E}_B \left(|\varphi\rangle_A \otimes |\chi\rangle_B \right)$$

$$= |\varphi'\rangle_A \otimes |\chi'\rangle_B$$

Example

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |011\rangle \right)$$

Is it entangled?

$A = \{1\}$ $B = \{2,3\}$

$$|\psi\rangle_{AB} = |0\rangle_A \otimes \frac{1}{\sqrt{2}}\left(|00\rangle_B + |11\rangle_B \right)$$ No

$A = \{1, 2\}$ $B = \{3\}$

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_A \otimes |0\rangle_B + |01\rangle_A \otimes |1\rangle_B \right)$$ Yes

Quantifying Entanglement

Entanglement measure

$E : D(\mathcal{H}) \mapsto \mathbb{R}$ s.t.

$$E(|\varphi\rangle\langle\varphi| \otimes |\chi\rangle\langle\chi|) = 0$$

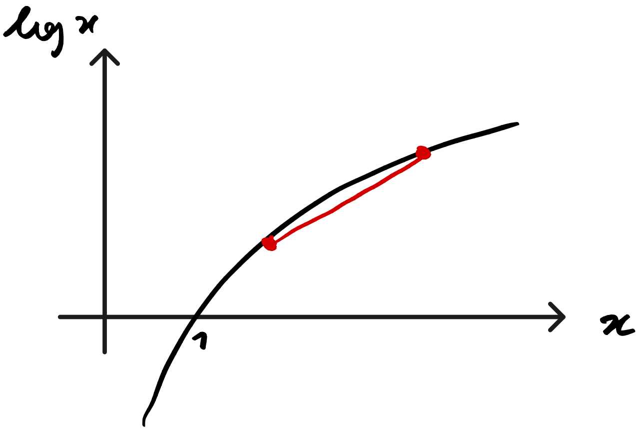
$$E(\hat{U}\otimes\hat{V}\,\hat{\rho}\,\hat{U}^{+}\otimes\hat{V}^{+}) = E(\hat{\rho})$$

Def von Neumann entropy

$$\boxed{\begin{aligned} S(\hat{\rho}) &= -\mathrm{Tr}(\hat{\rho}\log\hat{\rho}) \\ &= -\sum_{i=1}^{d} \lambda_i \log \lambda_i \end{aligned}}$$

$$\hat{\rho} = \sum_{i=1}^{d} \lambda_i |\varphi_i\rangle\langle\varphi_i|$$ spectral decomp.

$\log(x)$ is a concave function



$$-\sum_{i=1}^{d} \lambda_i \log(\lambda_i) = \sum_{i=1}^{d} \lambda_i \log\left(\frac{1}{\lambda_i}\right)$$

Jensen's inequality for concave functions

$$\leq \log\left(\sum_{i=1}^{d} \lambda_i \frac{1}{\lambda_i}\right)$$

$$= \log(d)$$

$$\Rightarrow \quad S(\hat{\rho}) \leq \log(d)$$

$S(\mathrm{Tr}_A(\hat{\rho})) = \log(d)$ iff $\hat{\rho}$ is maximally entangled

$$E(\hat{\rho}) = S(\mathrm{Tr}_A(\hat{\rho}))$$

Ex

$$\hat{\rho} = |\phi^+\rangle\langle\phi^+|$$

$$\hat{\rho}_A = \mathrm{Tr}_A(\hat{\rho}) = \frac{1}{2} I$$

$$S(\hat{\rho}_A) = -\frac{1}{2}\log\left(\frac{1}{2}\right) \times 2$$

$$= \log(2)$$

Maximal

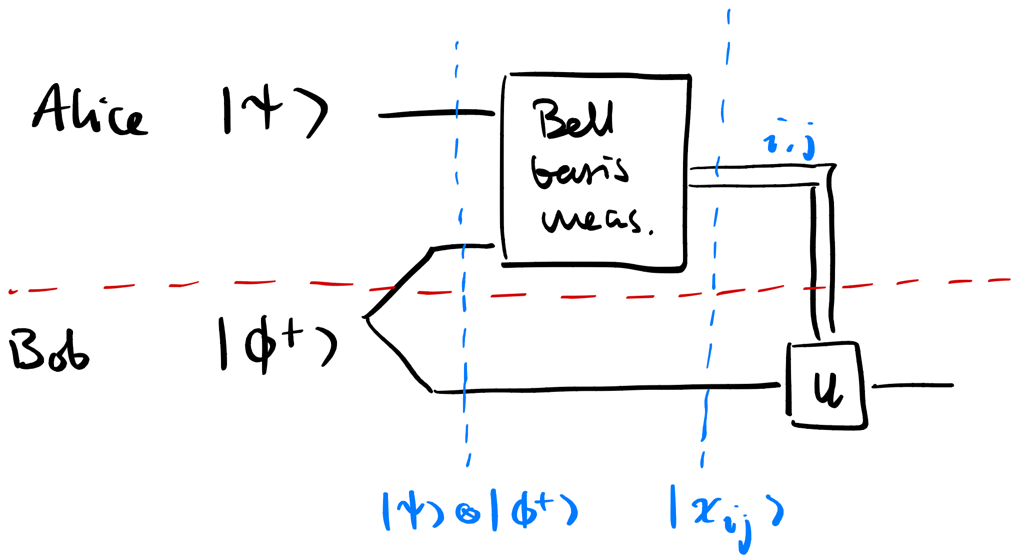
For a pure state

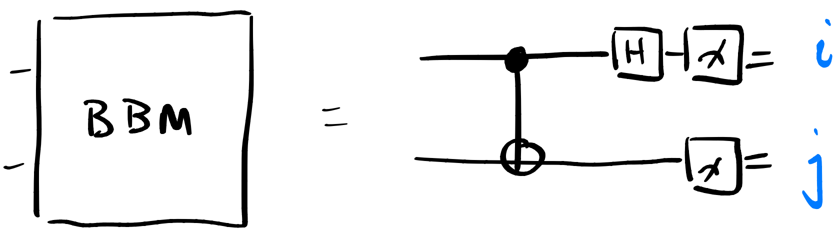
$$\hat{\rho} = |\psi\rangle\langle\psi|$$

$$\boxed{S(\hat{\rho}) = -\log(1) = 0}$$

Uses of entanglement

Quantum teleportation





Procedure

00 $(|\phi^+\rangle)$: $U = \hat{I}$

10 $(|\phi^-\rangle)$: $U = \hat{Z}$

$01 \quad (|\psi^+\rangle) : \quad \hat{U} = \hat{X}$

$11 \quad (|\psi^-\rangle) : \quad \hat{U} = \hat{Z}\hat{X}$

Alice measures in the Bell basis, sends two bits to Bob, who applies the recovery $\hat{U}$.

Initial state

$$|\psi\rangle \otimes |\phi^+\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes \left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)$$

$$\xmapsto{CNOT_{1\to 2}} \frac{\alpha}{\sqrt{2}}\left(|000\rangle + |011\rangle\right) + \frac{\beta}{\sqrt{2}}\left(|110\rangle + |101\rangle\right)$$

$$\xmapsto{H_1} \frac{\alpha}{2}\left(\begin{array}{l}|000\rangle + |100\rangle \\ + |011\rangle + |111\rangle\end{array}\right) + \frac{\beta}{2}\left(\begin{array}{l}|010\rangle - |110\rangle \\ + |001\rangle - |101\rangle\end{array}\right)$$

Measure, get $(i,j) = (0,0)$

$$(\langle 00| \otimes \hat{I}) \frac{\alpha}{2}\left(\begin{array}{l}|000\rangle + |100\rangle \\ + |011\rangle + |111\rangle\end{array}\right) + \frac{\beta}{2}\left(\begin{array}{l}|010\rangle - |110\rangle \\ + |001\rangle - |101\rangle\end{array}\right)$$

$$= \frac{\alpha}{2}|0\rangle + \frac{\beta}{2}|1\rangle$$

$$\Pr((i,j)=(0,0)) = \left\| \frac{\alpha}{2}|0\rangle + \frac{\beta}{2}|1\rangle \right\|^2$$

$$= \frac{1}{4}\left(|\alpha|^2 + |\beta|^2\right) = \frac{1}{4}$$

$$|\chi_{00}\rangle = \alpha|0\rangle + \beta|1\rangle$$

Measure, get $\underline{(i,j) = (1,0)}$

$$\left(\langle\phi^-| \otimes \hat{I}\right)|\psi\rangle \otimes |\phi^+\rangle$$

$$= \frac{1}{2}\left(\left(\langle 00| - \langle 11|\right) \otimes \hat{I}\right)\left(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\right)$$

$$= \frac{\alpha}{2}|0\rangle - \frac{\beta}{2}|1\rangle$$

$$|\chi_{10}\rangle = \alpha|0\rangle - \beta|1\rangle = Z|\chi_{00}\rangle$$

Measure, get $\underline{(i,j) = (0,1)}$

$$(\langle 10| \otimes \hat{I}) \frac{\alpha}{2}\left(\begin{array}{l} |000\rangle + |100\rangle \\ + |011\rangle + |111\rangle \end{array} \right) + \frac{\beta}{2}\left(\begin{array}{l} |010\rangle - |110\rangle \\ + |001\rangle - |101\rangle \end{array} \right)$$

$$= \frac{\alpha}{2}|1\rangle + \frac{\beta}{2}|0\rangle$$

$$|\chi_{01}\rangle = \alpha|1\rangle + \beta|0\rangle = \hat{X}|\chi_{00}\rangle$$

Measure, get $\underline{(i,j) = (1,1)}$

$$(\langle \psi^-| \otimes \hat{I})\, |\psi\rangle \otimes |\phi^+\rangle$$

$$= \frac{\alpha}{2}|1\rangle - \frac{\beta}{2}|0\rangle$$

$$|\chi_{11}\rangle = \alpha|1\rangle - \beta|0\rangle = \hat{X}\hat{Z}|\chi_{00}\rangle$$

$$|\chi_{ij}\rangle = \hat{X}^j \hat{Z}^i |\chi_{00}\rangle = \hat{U}_{ij} |\chi_{00}\rangle$$

$$\hat{U}_{ij} |\chi_{ij}\rangle = \hat{Z}^i \hat{X}^j (\hat{X}^j \hat{Z}^i) |\chi_{00}\rangle$$

$$= |\chi_{00}\rangle = |\psi\rangle$$

- Q. teleportation is not instantaneous

  classical communication limited by the speed of light.

Also extends to mixed states

Initial state $\hat{\rho} \otimes |\phi^+\rangle\langle\phi^+|$

$$= \sum_{i,j} \rho_{ij} \, \underset{A_1}{|i\rangle\langle j|} \otimes \frac{1}{2} \sum_{k,l} \underset{A_2}{|k\rangle\langle l|} \otimes \underset{B}{|k\rangle\langle l|}$$

$${}_A\langle\phi^+| \otimes \hat{I}_B \, | \hat{\rho} \otimes |\phi^+\rangle\langle\phi^+| \, |\phi^+\rangle_A \otimes \hat{I}_B$$

$$= \frac{1}{4}\left[\left({}_A\langle 00| + {}_A\langle 11| \right) \otimes \hat{I}_B \right]$$

$$\left[\sum_{i,j} \rho_{ij} \, |i\rangle\langle j|_{A_1} \otimes \frac{1}{2} \sum_{k,l} |k\rangle\langle l|_{A_2} \otimes |k\rangle\langle l|_{B} \right]$$

$$\left[\left(|00\rangle_A + |11\rangle_A \right) \otimes \hat{I}_B \right]$$

$$= \frac{1}{4} \sum_{i,j} \rho_{ij} \, \delta_{ik} \, \delta_{jl} \, |k\rangle\langle l|_B$$

$$= \frac{1}{4} \sum_{k,l} \rho_{kl} \, |k\rangle\langle l|_B = \frac{1}{4} \hat{\rho}$$

$$\mathrm{Tr}\left(\quad \swarrow \quad \right) = \frac{1}{4} \mathrm{Tr}(\hat{\rho}) = \frac{1}{4}$$

Repeat calculation for other Bell states:

$$\hat{\chi}_{ij} = \hat{X}^i \hat{Z}^i \hat{\rho} \, \hat{Z}^i \hat{X}^i$$

Return to $\hat{\rho}$ by applying $\hat{U}_{ij} = \hat{Z}^i \hat{X}^j$

If Bob does not receive the classical information then he would describe the state as

$$\frac{1}{4}\left(\hat{\rho} + \hat{X}\hat{\rho}\hat{X} + \hat{Y}\hat{\rho}\hat{Y} + \hat{Z}\hat{\rho}\hat{Z}\right)$$

$$= \frac{1}{2}\hat{I}$$

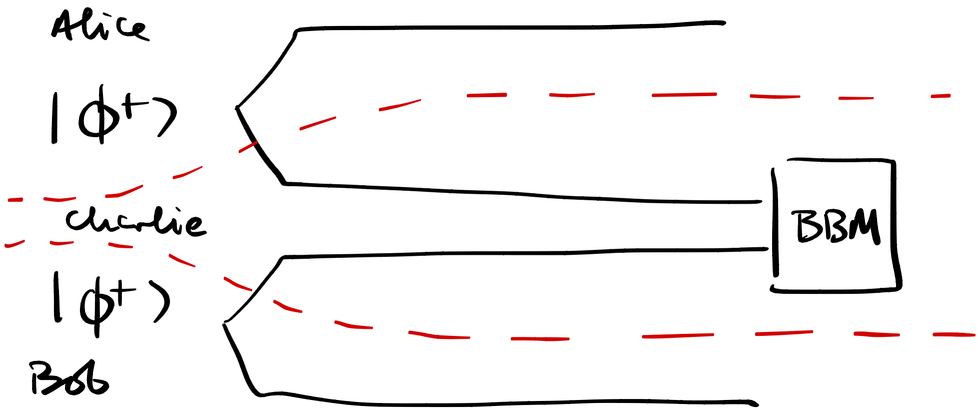
no information

$\ulcorner \hat{X}\hat{Z} = -i\hat{Y} \lrcorner$

$\hat{X}\hat{Y} = i\hat{Z}$

$\hat{Y} = i\hat{X}\hat{Z}$

Entanglement swapping



Initial state

$$|\phi^+\rangle \otimes |\phi^+\rangle$$

Measure in Bell basis, outcome (0,0)

$$\hat{I}_A \otimes {}_C\langle \phi^+ | \otimes \hat{I}_B \quad |\phi^+\rangle_{AC_1} \otimes |\phi^+\rangle_{C_2B}$$

$$= \frac{1}{2\sqrt{2}} \left(\hat{I}_B \otimes \left({}_C\langle 00| + {}_C\langle 11| \right) \otimes \hat{I}_B \right)$$

$$\left(\sum_{i,j} |i\rangle_A |ij\rangle_C |j\rangle_B \right)$$

$$= \frac{1}{2\sqrt{2}} \sum_{i,j} \delta_{ij} |i\rangle_A |j\rangle_B$$

$$= \frac{1}{2\sqrt{2}} \left(|00\rangle_{AB} + |11\rangle_{AB} \right)$$

$$= |\tilde{\chi}_{00}\rangle$$

$$\langle \tilde{\chi}_{00} | \tilde{\chi}_{00} \rangle = \| |\chi_{00}\rangle \|^2$$

$$= \frac{1}{8} \cdot 2 = \frac{1}{4} = \Pr\left((i,j) = (0,0) \right)$$

$$|\tilde{\chi}_{00}\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle_{AB} + |11\rangle_{AB} \right)$$

$$= |\phi^+\rangle_{AB}$$

Outcome (1, 1)

$$\hat{I}_A \otimes {}_C\langle \psi^- | \otimes \hat{I}_B \quad |\phi^+\rangle_{AC_1} \otimes |\phi^+\rangle_{C_2 B}$$

$$= \frac{1}{2\sqrt{2}} \left(\hat{I}_B \otimes \left({}_C\langle 01| - {}_C\langle 10| \right) \otimes \hat{I}_B \right)$$

$$\left(\sum_{i,j} |i\rangle_A |ij\rangle_C |j\rangle_B \right)$$

$$= \frac{1}{2\sqrt{2}} \left(|01\rangle_{AC} - |10\rangle_{AC} \right) = \frac{1}{2} |\psi^-\rangle_{AC}$$

So the output state is

i j			
$(0,0)$	$\vert\phi^+\rangle_{AC}$		
$(0,1)$	$\vert\psi^+\rangle_{AC}$	$=$	$\hat{X}\hat{I}\,\vert\phi^+\rangle_{AC}$
$(1,0)$	$\vert\phi^-\rangle_{AC}$	$=$	$\hat{Z}\hat{I}\,\vert\phi^+\rangle_{AC}$
$(1,1)$	$\vert\psi^-\rangle_{AC}$	$=$	$\hat{Z}\hat{I}\hat{X}\hat{I}\,\vert\phi^+\rangle_{AC}$

Charlie sends (i,j) to Alice (or Bob) who then applies a Pauli operator to recover $|\phi^+\rangle_{AC}$

Entangled state prepared on AC w/ no entangling operations applied between A & C!

Entangling operations such as CNOT can create entanglement

$$\text{CNOT}\,|x\rangle|y\rangle = |x\rangle|x+y \bmod 2\rangle$$

$x, y \in \{0,1\}$

$$\text{CNOT}\,|+\rangle\otimes|0\rangle = \text{CNOT}\left(\frac{1}{\sqrt{2}}\left[|0\rangle\otimes|0\rangle + |1\rangle\otimes|0\rangle\right]\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle\right)$$

$$= |\phi^+\rangle$$

Local operations ($\mathcal{E}_A \otimes I_A$, $I_A \otimes \mathcal{E}_B$) and classical communication (LOCC) cannot create entanglement.

Entanglement for mixed states

$$\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$$

Separable if it can be prepared by LOCC, entangled otherwise.

$$\hat{\rho}_{AB} = \sum_i p_i \hat{\rho}_i^A \otimes \hat{\rho}_i^B \qquad \text{Separable}$$

How to quantify entanglement?

Entropy doesn't work :(

$$\hat{\rho}_{AB} = \frac{1}{2}\hat{I}_A \otimes \frac{1}{2}\hat{I}_B \qquad \text{separable}$$

$$\hat{\rho}_A = \frac{1}{2}\hat{I}$$

$$S(\hat{\rho}_A) = S\left(\frac{1}{2}\hat{I}\right) = \log 2$$ maximal

It gets worse, as deciding whether a state is separable or entangled is NP-hard!

Ex $$\hat{\rho} = \frac{1}{2}|\phi^+\rangle\langle\phi^+| + \frac{1}{2}|\phi^-\rangle\langle\phi^-|$$

Separable?

$$\hat{\rho} = \frac{1}{4}\Big(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|$$

$$+ |00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|\Big)$$

$$= \frac{1}{2}\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big)$$

$$\hat{\rho} = \frac{1}{4}\left(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-| \right)$$

$$= \frac{1}{4}\hat{I} \quad !$$

Ex $\hat{\rho}_1$ $\hat{\rho}_2$ separable

$$\hat{\rho} = x\hat{\rho}_1 + (1-x)\hat{\rho}_2 \qquad x \in [0,1]$$

Separable (w/ prob x prepare a separable state, w/ prob $(1-x)$ prepare a separable state)

PPT criterion (Positive partial transpose)

Separable $\Rightarrow$ PPT

Partial transpose $(I_A \otimes T_B)\rho_{AB}$

If ρ_{AB} is separable then

$(I_A \otimes T_B)\rho_{AB}$ is non-negative.

But PPT $\not\Rightarrow$ Separable

PPT is necessary but not sufficient

$$\hat{\rho}_{AB} = \sum_i p_i \, \hat{\sigma}_i^A \otimes \hat{\tau}_i^B$$

$$\hat{I}_A \otimes \hat{T}_B \, \hat{\rho}_{AB} = \sum_i p_i \, \hat{\sigma}_i^A \otimes (\hat{\tau}_i^B)^T$$

Transpose preserves eigenvalues

τ_i^B PSD $\Rightarrow$ $(\tau_i^B)^T$ PSD

PPT also sufficient for

$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ & $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^3$

But for other $\mathcal{H}$

there exist entangled states

w/ positive partial transpose,

known as <u>bound entangled states</u>.

Werner state

$$\hat{\rho} = p\,|\Phi^+\rangle\langle\Phi^+| + \frac{(1-p)}{4}\hat{I}$$

$p \in [0, 1]$

$$\hat{\rho} = \begin{pmatrix} \frac{p}{2} + \frac{(1-p)}{4} & 0 & 0 & \frac{p}{2} \\ 0 & \frac{(1-p)}{4} & 0 & 0 \\ 0 & 0 & \frac{(1-p)}{4} & 0 \\ \frac{p}{2} & 0 & 0 & \frac{p}{2} + \frac{(1-p)}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+p}{4} & & & p/2 \\ & \frac{1-p}{4} & & \\ & & \frac{1-p}{4} & \\ p/2 & & & \frac{1+p}{4} \end{pmatrix}$$

$$\hat{I} \otimes \hat{T} \quad \hat{\rho}$$

$$= \begin{pmatrix} \frac{1+p}{4} & & & \\ & \frac{1-p}{4} & p/2 & \\ & p/2 & \frac{1-p}{4} & \\ & & & \frac{1+p}{4} \end{pmatrix}$$

Eigenvalues $\frac{1+p}{4} \quad \times 3 \qquad \frac{1-3p}{4}$

$$\begin{vmatrix} 1-p-\lambda & 2p \\ 2p & 1-p-\lambda \end{vmatrix} = 0$$

$$(1-p-\lambda)^2 = (2p)^2$$

$$1-p-\lambda = \pm 2p$$

$$\lambda = 1-p \pm 2p$$

$$\lambda_+ = 1+p \qquad \lambda_- = 1-3p$$

$\hat{\rho}$ has PPT for

$$\frac{1-3p}{4} \geq 0$$

$$p \leq \frac{1}{3}$$

For $p > 1/3$ the Werner state

is entangled.

CHSH inequality

(Clauser Horne Shimony Holt)

Assumptions

1. Realism

One can associate definite values to all properties of a physical system, measurement merely reveals these properties.

2. Locality

The measurement outcome in region A is not affected by a space-like separated event happening in region B.

Recall space-like separation

$(x_A, t_A) \qquad (x_B, t_B)$

$|x_A - x_B| > c\,|t_A - t_B|$

random variables

Alice measures one of two observables A_1 or A_2

Bob " " " B_1 or B_2

w/ outcomes $\in \{\pm 1\}$

Correlation function

$$C = A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2$$

$A_1 = a_1,\ A_2 = a_2,\ B_1 = b_1,\ B_2 = b_2$

$$\langle C \rangle = \sum_{a_1, a_2, b_1, b_2} p(A_i = a_i,\ B_i = b_i)$$

exp. value

$$(a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2)$$

$$a_1(b_1 + b_2) + a_2(b_1 - b_2)$$

a_1, a_2, b_1, b_2 $\quad a_1(b_1+b_2) + a_2(b_1-b_2)$

$1 \quad 1 \quad 1 \quad 1 \qquad 2$

$1 \quad 1 \quad 1 \; -1 \qquad 2$

Local realism

Simultaneous assignment of a_1, a_2, b_1, b_2

Max value is 2

$$\langle C \rangle = \sum_{a_1, a_2, b_1, b_2} p(A_i = a_i, B_i = b_i)(a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2)$$

$$\leq 2 \sum_{a_1, a_2, b_1, b_2} p(A_i = a_i, B_i = b_i)$$

$$= 2$$

$$\boxed{\langle C \rangle = \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle B_1 B_2 \rangle \leq 2}$$

We can estimate e.g. $\langle A_1 B_1 \rangle$ by measuring $A_1 B_1$ N times and averaging the outcomes.

Now consider

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$\hat{A}_1 = \hat{Z} \otimes \hat{I} \qquad \hat{B}_1 = \frac{1}{\sqrt{2}} \hat{I} \otimes (\hat{X} + \hat{Z})$$

$$\hat{A}_2 = \hat{X} \otimes \hat{I} \qquad \hat{B}_2 = \frac{1}{\sqrt{2}} \hat{I} \otimes (\hat{Z} - \hat{X})$$

$$\langle \hat{A}_1 \otimes \hat{B}_1 \rangle$$

$$= \langle \phi^+ | \hat{A} \otimes \hat{B} | \phi^+ \rangle$$

$$= \frac{1}{2\sqrt{2}} \left(\langle 00| + \langle 11| \right) \hat{Z} \otimes \hat{X} + \hat{Z} \otimes \hat{Z}$$

$$\left(|00\rangle + |11\rangle \right)$$

$$\hat{Z} \otimes \hat{Z} \left(|00\rangle + |11\rangle \right) = |00\rangle + |11\rangle$$

$$\hat{Z} \otimes \hat{X} \left(|00\rangle + |11\rangle \right) = |01\rangle - |10\rangle$$

$$\Rightarrow \quad \langle \phi^+ | \hat{A}_1 \otimes \hat{B}_1 | \phi^+ \rangle$$

$$= \frac{1}{2\sqrt{2}} \Big(\langle 00| + \langle 11| \Big) \Big(|00\rangle + |11\rangle + |01\rangle - |10\rangle \Big)$$

$$= \frac{1}{\sqrt{2}}$$

Note that $|\phi^+\rangle$ is the $+1$ eigenstate of $\hat{Z} \otimes \hat{Z}$ & $\hat{X} \otimes \hat{X}$,

$$\hat{Z} \otimes \hat{X} |\phi^+\rangle = |\psi^-\rangle$$

$$\hat{X} \otimes \hat{Z} |\phi^+\rangle = -1 |\psi^-\rangle$$

Bell states are an orthonormal basis $\langle \phi^+ | \psi^- \rangle = 0$

$$\langle \hat{A}_2 \otimes \hat{B}_1 \rangle$$

$$= \frac{1}{\sqrt{2}} \langle \phi^+ | \, \hat{X} \otimes \hat{X} + \hat{X} \otimes \hat{Z} \, | \phi^+ \rangle$$

$$= \frac{1}{\sqrt{2}} \langle \phi^+ | \phi^+ \rangle$$

$$\langle \hat{A}_1 \otimes \hat{B}_2 \rangle$$

$$= \frac{1}{\sqrt{2}} \langle \phi^+ | \, \hat{Z} \otimes \hat{Z} - \hat{Z} \otimes \hat{X} \, | \phi^+ \rangle$$

$$= \frac{1}{\sqrt{2}}$$

$$\langle \hat{A}_2 \otimes \hat{B}_2 \rangle$$

$$= \frac{1}{\sqrt{2}} \langle \phi^+ | \, \hat{X} \otimes \hat{Z} - \hat{X} \otimes \hat{X} \, | \phi^+ \rangle$$

$$= -\frac{1}{\sqrt{2}}$$

$$\langle \hat{A}_1 \hat{B}_1 \rangle + \langle \hat{A}_1 \hat{B}_2 \rangle + \langle \hat{A}_2 \hat{B}_1 \rangle - \langle \hat{A}_2 \hat{B}_2 \rangle$$

$$= \frac{4}{\sqrt{2}} = \frac{(\sqrt{2})^4}{\sqrt{2}} = 2\sqrt{2} > 2 \; !$$

Verified by experiments (Aspect)

$\Rightarrow$ Our world is not both local and realistic !

Which one would you give up out of the two?