Lecture 4 : Quantum Channels
Part II

Recap : Q. channel is a
completely - positive
trace - preserving
(linear) map (CPTP map)

CP $\left(\mathcal{E}_A \otimes \mathcal{I}_B \right) \hat{\rho}_{AB} \geq 0$

for any choice of B

TP $\mathrm{Tr}\left(\mathcal{E}(\hat{\rho}) \right) = \mathrm{Tr}(\hat{\rho})$

Linear $\mathcal{E}(\alpha \hat{\rho} + \beta \hat{\sigma})$

$= \alpha \mathcal{E}(\hat{\rho}) + \beta \mathcal{E}(\hat{\sigma})$

Kraus representation

$$\mathcal{E}(\hat{\rho}) = \sum_i \hat{K}_i \hat{\rho} \hat{K}_i^{\dagger}$$

where $\sum_i \hat{K}_i^{\dagger} \hat{K}_i = \hat{I}$

Today we will prove that every q. channel has a Kraus decomposition.

Theorem

$\mathcal{E}$ is a quantum channel iff it has a Kraus rep.

Proof

$\Leftarrow$

$$\hat{K}_i \left(\alpha \hat{\rho} + \beta \hat{\sigma} \right) \hat{K}_i^{+}$$

$$= \alpha \; \hat{K}_i \hat{\rho} \hat{K}_i^{+} + \beta \hat{K}_i \hat{\sigma} \hat{K}_i^{+}$$ Linear

$$\mathrm{Tr}\left(\sum_i \hat{K}_i \hat{\rho} \hat{K}_i^{+} \right)$$

$$= \sum_i \mathrm{Tr}\left(\hat{K}_i \hat{\rho} \hat{K}_i^{+} \right)$$

$$= \sum_i \mathrm{Tr}\left(\hat{\rho} \hat{K}_i \hat{K}_i^{+} \right)$$

$$= \mathrm{Tr}\left(\hat{\rho} \sum_i \hat{K}_i \hat{K}_i^{+} \right) = \mathrm{Tr}\left(\hat{\rho} \right)$$ TP

$$|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$$

$${}_{AB}\langle \psi | \left(\mathcal{E}_A \otimes I_B \right) \hat{\rho}_{AB} | \psi \rangle_{AB}$$

$$= \sum_i {}_{AB}\langle \psi | (\hat{K}_i \otimes \hat{I}_B) \hat{\rho} (K_i \otimes I_B)^\dagger | \psi \rangle_{AB}$$

$$= \sum_i {}_{AB}\langle \varphi | \hat{\rho} | \varphi \rangle_{AB} \geq 0$$

$$|\varphi\rangle_{AB} = (\hat{K}_i \otimes \hat{I}_B)^\dagger |\psi\rangle_{AB}$$

$\Rightarrow$

Choose $\mathcal{H}_B$ s.t. $\dim \mathcal{H}_A = \dim \mathcal{H}_B$
$= N$

Define

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle_A \otimes |i\rangle_B$$

$$(\mathcal{E}_A \otimes I_B) |\phi\rangle\langle\phi|_{AB} = \hat{\rho}'$$

Spectral decomp

$$= \sum_{\mu=1}^{N^2} \lambda_\mu |v_\mu\rangle\langle v_\mu|_{AB}$$

Consider generic

orthonormal

$$|\psi\rangle_A = \sum_{i=1}^{N} c_i \, |i\rangle_A$$

Define

$$|\psi^*\rangle_B = \sum_{i=1}^{N} c_i^* \, |i\rangle_B$$

$\forall \; |v_\mu\rangle$ define the linear operator

$$\hat{K}_\mu : |\psi\rangle_A$$

$$\mapsto \sum_{i=1}^{N} \sqrt{\lambda_\mu N} \; {}_A\langle i| \; {}_B\langle \psi^*| \; |v_\mu\rangle_{AB} \; |i\rangle_A$$

$$\alpha |\psi_1\rangle + \beta |\psi_2\rangle$$

$$\mapsto \sum \sqrt{\cdots} \; {}_A\langle i| \left(\alpha \, {}_B\langle \psi_1^*| + \beta \, {}_B\langle \psi_2^*| \right)$$

$$\rangle_{AB} |i\rangle_A$$

$${}_A\langle i' | \hat{K}_\mu | \psi \rangle_A$$

$$= \sqrt{\lambda_\mu N} \; {}_A\langle i' | \; {}_B\langle \psi^* | \; | v_\mu \rangle_{AB}$$

$${}_A\langle i' | \sum_{\mu=1}^{N^2} K_\mu | \psi \rangle\langle \psi |_A \; K_\mu^\dagger | j' \rangle_A$$

$$= \sum_{\mu=1}^{N^2} \lambda_\mu N \; {}_A\langle i' | \; {}_B\langle \psi^* | \; | v_\mu \rangle\langle v_\mu |_{AB} \; | \psi^* \rangle_B \, | j' \rangle_A$$

$$\hat{\rho} = \sum_{\mu=1}^{N^2} \lambda_\mu | v_\mu \rangle\langle v_\mu |_{AB} = (\mathcal{E}_A \otimes I_B) | \phi \rangle\langle \phi |_{AB}$$

$$= N \; {}_A\langle i' | \; {}_B\langle \psi^* | (\mathcal{E}_A \otimes I_B) | \phi \rangle\langle \phi |_{AB} \; | \psi^* \rangle_B \, | j' \rangle_A$$

$$= \sum_{i,j=1}^{N} {}_A\langle i' | \; {}_B\langle \psi^* | (\mathcal{E}_A \otimes I_B)$$

$$| i \rangle\langle j |_A \otimes | i \rangle\langle j |_B \; | j' \rangle_A | \psi^* \rangle_B$$

$$= \sum_{i,j=1}^{N} {}_A\langle i' | \, \mathcal{E}\left(|i\rangle\langle j|_A \right) | j'\rangle$$

$$\cdot \underbrace{{}_B\langle \psi^* | i\rangle_B}_{c_i} \underbrace{\langle j | \psi^* \rangle_B}_{c_j^*}$$

$\mathcal{E}$ linear

$$\downarrow$$

$$= {}_A\langle i' | \, \mathcal{E}\left(\sum_{i,j=1}^{N} c_i c_j^* |i\rangle\langle j|_A \right) |j'\rangle_B$$

$$= {}_A\langle i' | \, \mathcal{E}\left(|\psi\rangle\langle\psi|_A \right) |j'\rangle_B$$

Kraus rep for pure states ✓

Mixed state = convex mixture of pure states

$\mathcal{E}$ is linear $\Rightarrow$ Kraus rep for mixed states

Last check

$$\mathrm{Tr}\left(|i\rangle_A\langle j| \right)$$

$$= \sum_{i'} {}_A\langle i'| \, |i\rangle_A\langle j| \, |i'\rangle_A$$

$$= \delta_{ij}$$

$$= {}_A\langle j| \, I_A \, |i\rangle_A$$

$$\mathrm{Tr}\left(|i\rangle_A\langle j| \right) = \mathrm{Tr}\left(\mathcal{E}\left(|i\rangle_A\langle j| \right) \right)$$

$$= \sum_{\mu=1}^{N^2} \mathrm{Tr}\left(\hat{K}_\mu^\dagger \hat{K}_\mu |i\rangle_A\langle j| \right)$$

$$= \sum_{i'=1}^{N} \sum_{\mu=1}^{N^2} {}_A\langle i'| \hat{K}_\mu^\dagger \hat{K}_\mu |i\rangle \underbrace{\langle j|i'\rangle}_{\delta_{i'j}}$$

$$= {}_A\langle j| \sum_{\mu=1}^{N^2} \hat{K}_\mu^\dagger \hat{K}_\mu |i\rangle_A \implies \sum_\mu \hat{K}_\mu^\dagger \hat{K}_\mu = \hat{I} \quad \square$$

Reversibility

$\hat{U}$ can always be inverted by $\hat{U}^\dagger$

What about $\mathcal{E}$?

$$\mathcal{E}_2 \circ \mathcal{E}_1 (|\psi\rangle\langle\psi|)$$

$$= \sum_{\mu, a} \hat{K}_\mu \hat{L}_a |\psi\rangle\langle\psi| \hat{L}_a^\dagger \hat{K}_\mu^\dagger$$

$$\overset{!}{=} |\psi\rangle\langle\psi|$$

$$\Rightarrow \quad \hat{K}_\mu \hat{L}_a = \lambda_{\mu a} \hat{I}$$

$$\hat{L}_b^\dagger \hat{L}_a = \hat{L}_b^\dagger \sum_\mu \hat{K}_\mu^\dagger \hat{K}_\mu \hat{L}_a$$

$$= \sum_\mu \lambda_{\mu b}^* \lambda_{\mu a} \hat{I}$$

$$= \beta_{ba} \hat{I}$$

$\beta_{aa} \in \mathbb{R}^+$ unless $\hat{L}_a = 0$

$\hat{K}_a$ is a $d \times d$ matrix

Polar decomposition

$$\hat{K}_a = \hat{U}_a \sqrt{\hat{K}_a^\dagger \hat{K}_a}$$

$$= \sqrt{\beta_{aa}}\, \hat{U}_a$$

$$\hat{M}_b^\dagger \hat{M}_a = \sqrt{\beta_{aa} \beta_{bb}}\, \hat{U}_b^\dagger \hat{U}_a = \beta_{ba} \hat{I}$$

$$\hat{U}_a = \frac{\beta_{ba}}{\sqrt{\beta_{aa} \beta_{bb}}} \hat{U}_b$$

Each Kraus op is prop to the same unitary matrix ie evolution is unitary!

Thm: Stinespring dilation

For any quantum channel

$\mathcal{E}: D(\mathcal{H}_A) \to D(\mathcal{H}_A)$, we

can write it as

$$\mathrm{Tr}_B\left(\hat{U}_{AB} \left(\hat{\rho}_A \otimes |0\rangle\langle 0|_B \right) \hat{U}_{AB}^\dagger \right)$$

$$= \mathcal{E}(\hat{\rho}_A)$$

Proof

Define

Kraus ops of channel

$$\hat{U} : |\psi\rangle_A \mapsto \sum_\mu \hat{K}_\mu |\psi\rangle_A \otimes |v_\mu\rangle_B$$

orthonormal basis

$${}_A\langle\psi| \hat{U}^\dagger \hat{U} |\psi\rangle_A$$

$$= \sum_{\mu',\mu} {}_A\langle\psi| \otimes {}_B\langle v_{\mu'}| \hat{K}_{\mu'}^\dagger \hat{K}_\mu |\psi\rangle_A \otimes |v_\mu\rangle_B$$

$$= \sum_\mu {}_A\langle\psi| \hat{K}_\mu^\dagger \hat{K}_\mu |\psi\rangle_A$$

$$= {}_A\langle\psi|\psi\rangle_A = 1 \qquad \hat{U}^\dagger \hat{U} = \hat{I}_A$$

$\Rightarrow$ Can extend to a unitary on the full space

Such that

$$\hat{U}_{AB} : |\psi\rangle_A \otimes |0\rangle_B$$

$$\mapsto \sum_\mu \hat{K}_\mu |\psi\rangle_A \otimes |v_\mu\rangle_B$$

$\{|i\rangle_B\}$ $\{|v_\mu\rangle_B\}$ both orthonormal

$$\mathrm{Tr}_B \left(U_{AB} \left(\rho_A \otimes |0\rangle_B\langle 0| \right) U_{AB}^\dagger \right)$$

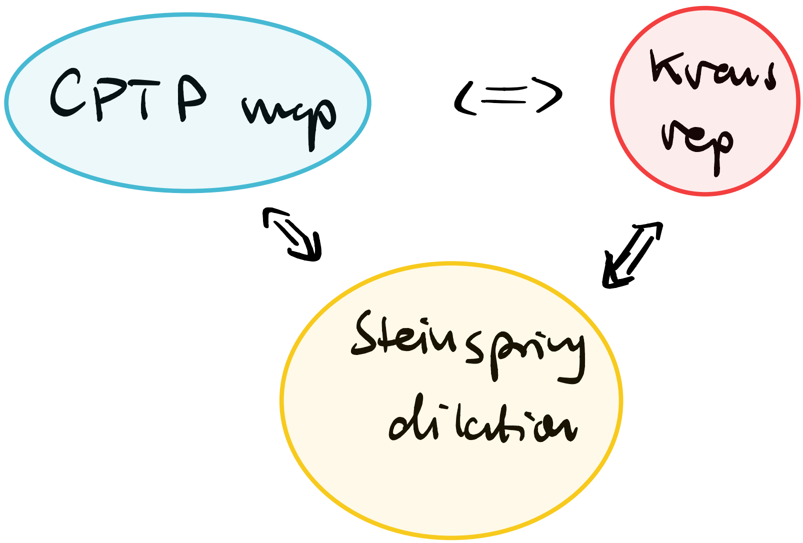
$$= \mathrm{Tr}_B \left(\sum_\mu \hat{K}_{\mu'} \hat{\rho}_A \hat{K}_\mu^\dagger \otimes |v_{\mu'}\rangle_B\langle v_\mu| \right)$$

$$= \sum_\mu \hat{K}_{\mu'} \hat{\rho}_A \hat{K}_\mu^\dagger \; \underbrace{\mathrm{Tr} \left(|v_{\mu'}\rangle\langle v_\mu| \right)}_{\delta_{\mu'\mu}}$$

$$= \sum_\mu \hat{K}_\mu \hat{\rho}_A \hat{K}_\mu^\dagger$$

$$= \mathcal{E}(\hat{\rho}_A) \qquad \square$$

Quantum Channels Recap



$\mathcal{E}$ quantum channel

CPTP Completely-positive trace-preserving

Kraus $\mathcal{E}(\rho) = \sum_i K_i \hat{\rho} K_i^\dagger$

Steinspring $\mathcal{E}(\rho) = \mathrm{Tr}_B \left(U_{AB}\, \rho_A \otimes |0\rangle\langle 0|_B \, U_{AB}^\dagger \right)$

Channel - state duality

Choi - Jamiolkowski isomorphism

$$\mathcal{J}(\mathcal{E}) = (\mathcal{E}_A \otimes I_B) |\phi\rangle\langle\phi|_{AB}$$

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle_A |i\rangle_B$$

$$\mathcal{J}^{-1}[\hat{\rho}_{AB}](\hat{\sigma}_A)$$

$$= N \, \mathrm{Tr}_B\left((\hat{I}_A \otimes \hat{\sigma}_B^T) \hat{\rho}_{AB} \right)$$

$$\mathcal{J}^{-1}[\mathcal{J}(\mathcal{E})](\hat{\sigma}_A)$$

$$= N \, \mathrm{Tr}_B\Big((\hat{I}_A \otimes \hat{\sigma}_B^T)$$

$$\mathcal{E}_A \otimes I_B |\phi\rangle\langle\phi|_{AB} \Big)$$

$$= \sum_{i,j} \mathrm{Tr}_B \Big(\big(\hat{I}_A \otimes \hat{\sigma}_B^T \big) \, \mathcal{E}\big(|i\rangle\langle j|_A \big) \otimes |i\rangle\langle j|_B \Big)$$

$$= \sum_{i,j} \mathcal{E}\big(|i\rangle\langle j|_A \big) \, \mathrm{Tr}\big(\hat{\sigma}_B^T \, |i\rangle\langle j|_B \big)$$

$\mathrm{Tr}_B\big(\hat{M}_A \otimes \hat{N}_B\big)$

$\hat{M}_A \otimes \mathrm{Tr}\big(\hat{N}_B\big)$

$\shortparallel$

$\langle j | \hat{\sigma}_B^T | i \rangle = \langle i | \hat{\sigma}_B | j \rangle$

$$= \sum_{i,j} \sigma_{ij} \, \mathcal{E}\big(|i\rangle\langle j|_A \big) = \mathcal{E}\big(\hat{\sigma}_A \big)$$

Homework to show

$$\mathcal{J}\big(\mathcal{J}^{-1}[\hat{\rho}_{AB}](\cdot) \big) = \hat{\rho}_{AB}$$

Freedom in Kraus reps

Consider the state rep. of a channel

$$\mathcal{J}(\mathcal{E}) = (\mathcal{E}_A \otimes I_B)\, |\phi\rangle\langle\phi|_{AB}$$

where $|\phi\rangle = \frac{1}{\sqrt{N}} \sum_i |i\rangle_A \otimes |i\rangle_B$

$$\mathcal{J}(\mathcal{E}) = \sum_\mu \hat{K}_\mu \otimes \hat{I}_B\, |\phi\rangle\langle\phi|_{AB}\, \hat{K}_\mu^\dagger \otimes \hat{I}_B$$

$$= \sum_\mu |\psi_\mu\rangle\langle\psi_\mu|_{AB}$$

But we can equally well express this state in a different basis

$$|\varphi_a\rangle_{AB} = \sum_\mu U_{a\mu} |\psi_\mu\rangle_{AB}$$

unitary $\hat{U}$

$$J(\mathcal{E}) = \sum_a |\varphi_a\rangle\langle\varphi_a|_{AB}$$

$$= \sum_a \sum_{\mu,\mu'} U_{a\mu} |\psi_\mu\rangle\langle\psi_{\mu'}|_{AB}\, U^*_{a\mu'}$$

$$= \sum_a \sum_{\mu,\mu'} U_{a\mu} (\hat{K}_\mu \otimes \hat{I}_B) |\phi\rangle\langle\phi|_{AB}$$

$$(\hat{K}_\mu^\dagger \otimes \hat{I}_B)\, U^*_{a\mu'}$$

Define $\hat{M}_a = \sum_\mu U_{a\mu} \hat{K}_\mu$

Kraus reps related by unitary matrix

$$= \hat{U}\hat{K}_\mu$$

$$\hat{M}_a^\dagger = \sum_\mu U^*_{a\mu'} \hat{K}_\mu^\dagger \quad \text{by def}$$

How to specify a q. channel

Hermitian matrix has

d real diagonals

$$2 \sum_{i=1}^{d-1} d = 2 \frac{d(d-1)}{2}$$ off diagonal real numbers

$$d + d^2 - d = d^2$$

Hermiticity preserving linear map

has $d^2 \times d^2$ parameters.

$\mathcal{E}$ is constrained to preserve

trace on all inputs (d^2 constraints)

$$\# \text{ params} = d^4 - d^2$$

1 qubit $16 - 4 = 12$ params

2 qubit $256 - 16 = 240$

4^4

Compare w/ # parameters to specify a density matrix

$\hat{\rho}$ Hermitian d^2

$\mathrm{Tr}(\hat{\rho}) = 1$ $d^2 - 1$

1 qubit 3 params

2 qubit 15

Pauli transfer matrix (PTM)

$\mathcal{E}$ q. channel on n-qubits $d = 2^n$

Define

$$(R_{\mathcal{E}})_{i,j} = \frac{1}{d} \mathrm{Tr}\left(\mathcal{E}(\hat{P}_i) \hat{P}_j \right)$$

$$\hat{P}_i = \hat{P}_{i_1} \otimes \hat{P}_{i_2} \otimes \dots \otimes \hat{P}_{i_n}$$

1q Pauli ops

E.g. $n = 1$

$$R_{\mathcal{E}} = \left(\begin{array}{c|c} 1 & 0\ 0\ 0 \\ \hline \underline{v} & M \end{array} \right)$$

$$\frac{1}{2}\mathrm{Tr}(\hat{I}) = 1$$

$$\frac{1}{2}\mathrm{Tr}(\hat{X}) = \frac{1}{2}\mathrm{Tr}(\hat{Y}) = \frac{1}{2}\mathrm{Tr}(\hat{Z})$$

$$= 0$$

In this formalism a density matrix is a vector

$$(\underline{r})_i = \mathrm{Tr}(\hat{P}_i \hat{\rho})$$

You will sometimes see this written as $|\rho\rangle\rangle = \underline{r}$

$$(\underline{r})_0 = 1 = \mathrm{Tr}(\hat{\rho})$$

Properties

- $\mathcal{E}_2 \circ \mathcal{E}_1$ channel composition

  $R_{\mathcal{E}_2} R_{\mathcal{E}_1}$ PTM multiplication

- $(R_{\mathcal{E}})_{i,j} \in [-1, 1]$

- For Clifford gates $R_{\mathcal{E}}$ is a permutation matrix

- For Pauli chanels the PTM is diagonal

Ex

$$\mathcal{E}_x(\hat{\rho}) = (1-p)\hat{\rho} + p\hat{X}\hat{\rho}\hat{X}$$

$$\mathrm{Tr}\left(\mathcal{E}_x(\hat{X})\,\hat{\rho}\right)$$

$$= \mathrm{Tr}\left(\hat{X}\hat{\rho}\right) = \begin{cases} 1 & \hat{\rho} = \hat{X} \\ 0 & \text{else} \end{cases}$$

$$\mathrm{Tr}\left(\mathcal{E}_x(\hat{y})\,\hat{\rho}\right)$$

$$= \mathrm{Tr}\left(\left((1-p)\hat{y} - p\hat{y}\right)\hat{\rho}\right)$$

$$= (1-2p)\,\mathrm{Tr}\left(\hat{y}\,\hat{\rho}\right) = \begin{cases} 1-2p & \hat{\rho} = \hat{y} \\ 0 & \text{else} \end{cases}$$

$$
R_{\varepsilon_x} = \begin{array}{c} \begin{array}{cccc} \hat{I} & \hat{X} & \hat{Y} & \hat{Z} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-2p & 0 \\ 0 & 0 & 0 & 1-2p \end{pmatrix} \end{array}
$$

$$
\hat{\rho} = \frac{1}{2}\hat{I}
$$

$$
|I\rangle\rangle = (1, 0, 0, 0)^T
$$

$$
R_\varepsilon |I\rangle\rangle = |I\rangle\rangle
$$

$$
|\rho\rangle\rangle = (1, r_x, r_y, r_z)
$$

$$
R_\varepsilon |\rho\rangle\rangle = (1, r_x, (1-2p) r_y, (1-2p) r_z)
$$