Lecture 3 Quantum Channels

How to describe unitary acting on a subsystem then tracing out the subsystem?

Def : Quantum Channel

$$\mathcal{E} : B(\mathcal{H}) \to B(\mathcal{H}) \quad \text{s.t.}$$

1. Linear $\mathcal{E}(\alpha \hat{\rho}_1 + \beta \hat{\rho}_2) =$
$$\alpha\, \mathcal{E}(\hat{\rho}_1) + \beta\, \mathcal{E}(\hat{\rho}_2)$$

2. Preserves Hermiticity
$$\hat{\rho} = \hat{\rho}^{\dagger} \Rightarrow \mathcal{E}(\hat{\rho}) = \mathcal{E}(\hat{\rho})^{\dagger}$$

3. Preserves trace
$$\mathrm{Tr}(\hat{\rho}) = \mathrm{Tr}(\mathcal{E}(\hat{\rho}))$$

4. Completely positive

any finite dim $\mathcal{H}_B$

$$\mathcal{E}_A \otimes I_B : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

$$\hat{\rho}_{AB} \geq 0 \implies (\mathcal{E}_A \otimes I_B)(\hat{\rho}_{AB}) \geq 0$$

all eigenvalues are +ve

$\equiv \langle \psi | \rho_{AB} | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}$

These conditions ensure that $\mathcal{E}$ maps density matrices to density matrices.

Condition 4 is a bit mysterious, why not just require

$$\hat{\sigma}_A \geq 0 \implies \mathcal{E}_A(\hat{\sigma}_A) \geq 0 \; ?$$

Consider the transpose map

$$T: |i\rangle\langle j| \mapsto |j\rangle\langle i|$$

$$T: \hat{\rho} \mapsto \hat{\rho}^T$$

$$\langle\psi| \hat{\rho}^T |\psi\rangle$$

$$= \sum_{i,j} \psi_j^* (\hat{\rho}^T)_{ji} \psi_i$$

$$= \sum_{i,j} \psi_i \hat{\rho}_{ij} \psi_j^*$$

$$= \langle\psi^*| \hat{\rho} |\psi^*\rangle \geq 0 \quad \text{as } \hat{\rho} \geq 0$$

T is positive

Now consider

$$|\tilde{\Phi}\rangle_{AB} = \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle_B$$

$$(T \otimes I)(|\tilde{\Phi}\rangle\langle\tilde{\Phi}|_{AB})$$

$$= (T \otimes I)\left(\sum_{i,j} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B\right)$$

$$= \sum_{i,j} |j\rangle\langle i|_A \otimes |i\rangle\langle j|_B$$

$$:= \mathrm{SWAP}_{AB}$$

$$\mathrm{SWAP}_{AB} : |\psi\rangle_A \otimes |\tau\rangle_B$$

$$= \sum_{i,j} \psi_i \tau_j \, |i\rangle_A \otimes |j\rangle_B$$

$$\mapsto \sum_{i,j} \tau_j \psi_i \, |j\rangle_A \otimes |i\rangle_B$$

$$= |\tau\rangle_A \otimes |\psi\rangle_B$$

For $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{I} & 0 \\ 0 & \hat{X} \end{pmatrix}$$

Eigvalues are $+1$ (multiplicity 3)

-1

$\Rightarrow$ SWAP is not positive

$$\Rightarrow \quad \underbrace{(T \otimes I)(|\tilde{\Phi}\rangle\langle\tilde{\Phi}|_{AB})}_{\text{not positive}}$$

$\Rightarrow$ Not a valid density matrix

T not completely positive

Linearity (if time)

Why should $\mathcal{E}$ be linear?

Suppose we prepare $\hat{\rho}_i$ w/ probability p_i and apply $\mathcal{E}$

Then w/ probability p_i we get $\mathcal{E}(\rho_i)$

Recall: ensemble interpretation of density matrices

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i$$

$$\mathcal{E}(\hat{\rho}) = \mathcal{E}\left(\sum_i p_i \hat{\rho}_i\right)$$

$$= \sum_i p_i \, \mathcal{E}(\hat{\rho}_i) \text{ by above argument.}$$

Nonlinear $\mathcal{E}$ w/ strange consequences

$$\mathcal{E}(\hat{\rho}) = e^{i\pi \hat{X}\,\mathrm{Tr}(\hat{X}\hat{\rho})}\,\hat{\rho}\,e^{-i\pi \hat{X}\,\mathrm{Tr}(\hat{X}\hat{\rho})}$$

$$\hat{\rho} = \frac{1}{2}\hat{I} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$\mathrm{Tr}(\hat{X}\hat{\rho}) = \mathrm{Tr}(\hat{X}) = 0$$

$$\mathcal{E}(\hat{\rho}) = \hat{\rho}$$

w/ prob 1/2 prep $|0\rangle$ and get $|0\rangle$

$$\hat{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\mathrm{Tr}(\hat{X}\hat{\rho}) = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \frac{1}{2}$$

$$\mathcal{E}(\hat{\rho}) = \hat{X}\hat{\rho}\hat{X}$$

w/ prob $\frac{1}{2}$ prep $|0\rangle$ and get $|1\rangle$, why does it depend on other half?!

Quantum channels are sometimes called

1) super operators

2) completely-positive trace-preserving maps (CPTP maps)

Kraus representation

Initial state

$$\hat{\rho}_{AB} = \hat{\sigma}_A \otimes |0\rangle\langle 0|_B$$

U_{AB} unitary evolution

$$\hat{\sigma}_A' = \mathrm{Tr}_B \left(U_{AB} \, \hat{\rho} \, U_{AB}^{\dagger} \right)$$

$$= \mathrm{Tr}_B \left(U_{AB} \left(\hat{\sigma}_A \otimes |0\rangle\langle 0|_B \right) U_{AB}^{\dagger} \right)$$

$$\mathrm{Tr}_B\left(U_{AB} \left(\hat{\sigma}_A \otimes |0\rangle\langle 0|_B \right) U_{AB}^\dagger \right)$$

$$= \sum_i {}_B\langle i| \, U_{AB} \left(\hat{\sigma}_A \otimes |0\rangle\langle 0|_B \right) U_{AB}^\dagger \, |i\rangle_B$$

orthonormal basis

Define $\hat{K}_i = {}_B\langle i| \, U_{AB} \, |0\rangle_B$

This is an operator acting on A

$$= \sum_i \hat{K}_i \, \hat{\sigma}_A \, \hat{K}_i^\dagger$$

Kraus operators

${}_A\langle \mu | K_i | \nu \rangle_A$

$= {}_A\langle \mu | \otimes {}_B\langle i | \, U_{AB} \, |\nu\rangle_A \otimes |0\rangle_B$

$$\mathcal{E}(\sigma_A) = \sum_i \hat{K}_i \, \hat{\sigma}_A \, \hat{K}_i^\dagger$$

Kraus or operator sum representation of the channel.

Are there any conditions on the $\hat{K}_i$?

$$\sum_i \hat{K}_i^\dagger \hat{K}_i$$

$$= \sum_i {}_B\langle 0| U_{AB}^\dagger |i\rangle\langle i|_B U_{AB} |0\rangle_B$$

Recall $\sum_i |i\rangle\langle i| = \hat{I}$

$$= \underbrace{{}_B\langle 0| U_{AB}^\dagger U_{AB} |0\rangle_B}_{1} \hat{I} = \hat{I}$$

$$= \sum_i \hat{K}_i^\dagger \hat{K}_i$$

We will show next time that every quantum channel has a Kraus representation.

Def Unitary channel has just one Kraus operator

$$\mathcal{E}(\hat{\rho}) = \hat{K}\hat{\rho}\hat{K}^{+}$$

$\hat{K}\hat{K}^{+} = \hat{I}$ ie $\hat{K}$ unitary

Def : Unital channel $\mathcal{E}$

$$\mathcal{E}(\hat{I}) = \hat{I}$$

equivalently

$$\sum_i \hat{K}_i \hat{K}_i^{+} = \sum_i \hat{K}_i^{+} \hat{K}_i = \hat{I}$$

Cannot increase purity

Examples of quantum channels

Depolarizing channel

$$\mathcal{E}_{depol}(\hat{\rho}) = (1-p)\hat{\rho} + p\frac{1}{2}\hat{I}$$

$p \in [0,1]$

Interpretation: random angle rotation about a random axis (in Bloch sphere)

Unital $\quad \mathcal{E}_{depol}(\hat{I}) = \hat{I}$

Equivalent definition

$$\mathcal{E}_{depol}(\hat{\rho}) = (1-p)\hat{\rho} + \frac{p}{3}\left(\hat{X}\hat{\rho}\hat{X} + \hat{Y}\hat{\rho}\hat{Y} + \hat{Z}\rho\hat{Z}\right)$$

Homework

Showing they are equivalent

Amplitude damping

$$\hat{K}_0^+ \hat{K}_0 = \begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}$$

$$\mathcal{E}_{damp}(\hat{\rho}) = \hat{K}_0 \hat{\rho} \hat{K}_0^+ + \hat{K}_1 \hat{\rho} \hat{K}_1^+$$

$$\hat{K}_0 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \qquad \hat{K}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$$

$p \in [0,1]$ damping parameter

Non unitary transformation modelling relaxation of a qubit to the ground state due to spontaneous emission.

Skip 2 pages!

$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1+r_3 & r_x + i r_y \\ r_x - i r_y & 1-r_3 \end{pmatrix}$$

$$\xmapsto{\mathcal{E}_{damp}} \frac{1}{2}\begin{pmatrix} 1+(p+r_3(1-p)) & \sqrt{1-p}\,(r_x+ir_y) \\ \sqrt{1-p}\,(r_x-ir_y) & 1-(p+r_3(1-p)) \end{pmatrix}$$

$$\underline{r}' = (\sqrt{1-p}\, r_x, \sqrt{1-p}\, r_y, \gamma + r_3(1-p))$$

$$\begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1+r_3 & r_x+ir_y \\ r_x-ir_y & 1-r_3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{p}\,(r_x-ir_y) & \sqrt{p}\,(1-r_3) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} p(1-r_3) & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}\begin{pmatrix} 1+r_3 & r_x+ir_y \\ r_x-ir_y & 1-r_3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$$

$$= \begin{pmatrix} 1+r_3 & r_x+ir_y \\ \sqrt{1-p}\,(r_x-ir_y) & \sqrt{1-p}\,(1-r_3) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$$

$$= \begin{pmatrix} 1+r_3 & \sqrt{1-p}\,(r_x+ir_y) \\ \sqrt{1-p}\,(r_x-ir_y) & (1-p)(1-r_3) \end{pmatrix}$$

Better to show them

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

$$\mathcal{E}_{damp}(\hat{\rho})$$

$$= \begin{pmatrix} \rho_{00} + p\,\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & (1-p)\,\rho_{11} \end{pmatrix}$$

Derive the same way as previous page but using ρ_{00} etc.

$$\underline{r}' = \left(\sqrt{1-p}\, r_x ,\ \sqrt{1-p}\, r_y ,\ p + r_3(1-p) \right)$$

$$\lim_{p \to 0} \underline{r}' = \left(r_x , r_y , r_3 \right) = \underline{r}$$

$$\lim_{p \to 1} \underline{r}' = \left(0, 0, 1 \right)$$

Not unital $\mathcal{E}_{damp}(\hat{I}) \neq \hat{I}$

Can increase the purity

$$\mathcal{E}_{damp}(\rho) \overset{p=1}{\longmapsto} |0\rangle\langle 0|$$

Dephasing channel

$$\mathcal{E}_Z(\hat{\rho}) = (1-p)\hat{\rho} + \hat{Z}\hat{\rho}\hat{Z} \quad p \in [0,1]$$

Coupling to the environment

$\rightarrow$ random fluctuations of qubit frequency, loss of info. about the coherence of the qubit

Random angle rotation around the z axis. Unital.

Bit flip channel

$$\mathcal{E}_X(\hat{\rho}) = (1-p)\hat{\rho} + \hat{X}\hat{\rho}\hat{X} \quad p \in [0,1]$$

Qubit T_1

$\mathcal{E}_{damp}$ discrete change of qubit state in time Δt,

Relaxation rate $\gamma = p/\Delta t$.

Probability of relaxation per unit time.

$$\mathcal{E}_{damp}^{\Delta t}(\hat{\rho}) = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & (1-p)\rho_{11} \end{pmatrix}$$

$t = n\Delta t$

$$\mathcal{E}_{damp}^{t}(\hat{\rho}) = \mathcal{E}_{damp}^{\Delta t} \circ \ldots \circ \mathcal{E}_{damp}^{\Delta t}(\hat{\rho})$$

$\rho_{00} + p\rho_{11} + p(1-p)\rho_{11}$

$\rho_{00} + p\rho_{11} + p(1-p)\rho_{11} + p(1-p)^2\rho_{11}$

$$1-(1-p)^n$$

$$1-\left(1-np+\binom{n}{2}p^2-\binom{n}{3}p^3+\cdots\right)$$

$$=np-\binom{n}{2}p^2+\binom{n}{3}p^3-\cdots$$

$n=3$

$$=3p-3p^2+p^3$$

$$\rho_{00}^{n}=\rho_{00}^{0}+\underbrace{\sum_{i=0}^{n-1}p(1-p)^i\rho_{11}}_{}$$

$$p\rho_{11}\left(\frac{1-(1-p)^n}{1-(1-p)}\right)$$

$$=\left(1-(1-p)^n\right)\rho_{11}$$

$$\mathcal{E}^{t}_{\text{damp}}(\hat{\rho}) = \mathcal{E}^{\Delta t}_{\text{damp}} \circ \dots \circ \mathcal{E}^{\Delta t}_{\text{damp}}(\hat{\rho})$$

$$= \begin{pmatrix} \rho_{00} + (1-(1-p)^n)\rho_{11} & (1-p)^{n/2}\rho_{01} \\ (1-p)^{n/2}\rho_{10} & (1-p)^n \rho_{11} \end{pmatrix}$$

$$(1-p)^n = (1-\gamma t/n)^n$$

$$\underset{n\to\infty}{=} e^{-\gamma t}$$

$$\log\left(1-\frac{\gamma t}{n}\right)^n = n \log\left(1-\frac{\gamma t}{n}\right)$$

$\log(1+x) \simeq x$ for x small

$$= n\left(-\frac{\gamma t}{n}\right) = -\gamma t$$

$$\begin{pmatrix} \rho_{00} + (1-(1-p)^n)\rho_{11} & (1-p)^{n/2}\rho_{01} \\ (1-p)^{n/2}\rho_{10} & (1-p)^n \rho_{11} \end{pmatrix}$$

$$\underset{n\to\infty}{=} \begin{pmatrix} \rho_{00} (1 - e^{-\gamma t})\rho_{11} & e^{-\gamma t/2}\rho_{01} \\ e^{-\gamma t/2}\rho_{10} & e^{-\gamma t}\rho_{11} \end{pmatrix}$$

$$\boxed{\text{Relaxation time } T_1 := 1/\gamma}$$

Transmon qubits

$T_1 \sim 1\mu s - 500\mu s$

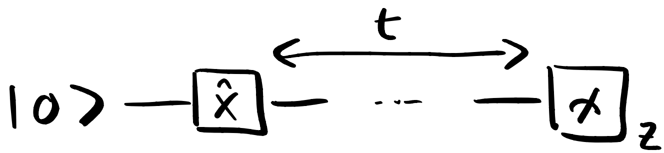
Characteristic time by which the qubit relaxes to the ground state.

Can use this for qubit reset

$$\mathcal{E}^t_{damp}(\hat{\rho}) \underset{t\to\infty}{=} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \forall \hat{\rho} \in D(\mathcal{H})$$

In practice $t \simeq 3T_1$ good enough

Measuring T_1



"π pulse"

$$\hat{\rho}(t) = \mathcal{E}^{t}_{damp}(|1\rangle\langle 1|)$$

$$= \begin{pmatrix} 1 - e^{-t/T_1} & 0 \\ 0 & e^{-t/T_1} \end{pmatrix}$$

Vary t, measure

$N_0(t)$ # shots measured $|0\rangle$

$N_1(t)$ # shots measured $|1\rangle$

$$p_1(t) = \frac{N_1(t)}{N_0(t) + N(t)}$$

Fit $p_1(t)$

$$= A + B e^{-t/c}$$

Qubit T_2

$\mathcal{E}_Z$ discrete change of qubit state in short time Δt

$$\hat{Z}\hat{\rho}\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{00} & \rho_{01} \\ -\rho_{10} & -\rho_{11} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \rho_{00} & -\rho_{01} \\ -\rho_{10} & \rho_{11} \end{pmatrix}$$

$$\mathcal{E}_Z^{\Delta t}(\hat{\rho}) = \begin{pmatrix} \rho_{00} & (1-2p)\rho_{01} \\ (1-2p)\rho_{10} & \rho_{11} \end{pmatrix}$$

Dephasing rate $\gamma_\phi = 2p/\Delta t$

Probability of $\hat{Z}$ error per unit time.

Dephasing time $\gamma_\phi = \frac{1}{T_\phi}$

After time $t = n\Delta t$

$$\mathcal{E}_z^t(\hat{\rho}) = \begin{pmatrix} \rho_{00} & e^{-t/T_\phi}\rho_{01} \\ e^{-t/T_\phi}\rho_{10} & \rho_{11} \end{pmatrix}$$

In real life we have amplitude damping and dephasing at the same time

Commute so order doesn't matter

$$\mathcal{E}_{damp}^t \circ \mathcal{E}_z^t(\hat{\rho})$$

$$= \begin{pmatrix} \rho_{00} - e^{-t/T_1}\rho_{11} & e^{-t/T_2}\rho_{01} \\ e^{-t/T_2}\rho_{10} & e^{-t/T_1}\rho_{11} \end{pmatrix}$$

where $T_2 = \left(\frac{1}{T_\phi} + \frac{1}{2T_1} \right)^{-1}$

coherence time

$T_2 \sim 1\,\mu s - 200\,\mu s$ transmon

$0 \lesssim T_2 \lesssim 2T_1$

T_1 limited if $T_2 \approx 2T_1$

Measuring T_2

Ramsey experiment

$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ "$\pi/2$ pulse"

$$\hat{\rho}(t) = \mathcal{E}_{damp}^{t} \circ \mathcal{E}_{z}^{t} (\hat{\rho})$$

$$= \frac{1}{2} \begin{pmatrix} 1 - e^{-t/T_2} & * \\ * & 1 - e^{-t/T_2} \end{pmatrix}$$

Vary t, measure

$N_0(t)$ # shots measured $|0\rangle$

$N_1(t)$ # shots measured $|1\rangle$

$$p_0(t) = \frac{N_0(t)}{N_0(t) + N(t)}$$

Fit $p_0(t)$

$$= A + B e^{-t/c}$$