Lecture 2: Quantum measurements

Recall that an <u>observable</u> in Quantum Mechanics is a <u>Hermitian operator</u> $\hat{A}$.

As we showed last time, this implies that $\hat{A}$ is diagonalizable and has real eigenvalues.

$$\hat{A} = \sum_a \lambda_a \hat{P}_a$$

$\{\lambda_a\}$ distinct eigenvectors of $\hat{A}$

eigenvalue

Projector onto eigenspace a

We have the properties

$\hat{P}_a^{\dagger} = \hat{P}_a$ Hermitian

$\hat{P}_a \hat{P}_b = 0$ Orthogonal

$$\hat{P}_a^2 = \hat{P}_a \qquad \text{Projector}$$

$$\sum_a \hat{P}_a = \hat{I} \qquad \text{Complete}$$

This is a consequence of the fact that $\mathcal{H} = \bigoplus_a \mathcal{H}_a$ ← eigenspaces

$$|\psi\rangle = \sum_a |\psi_a\rangle = \underbrace{\sum_a \hat{P}_a}_{\hat{I}} |\psi\rangle$$

But this is essentially the <u>spectral theorem</u>.

Let $\hat{\rho} \in D(\mathcal{H})$ be a state, suppose we measure $\hat{A}$.

The probability of outcome i is $\boxed{\text{Prob}(a) = \text{Tr}(\hat{\rho}\,\hat{P}_a)}$ and the corresponding post-measurement state is

$$\boxed{\hat{\rho}_a = \frac{\hat{P}_a\,\hat{\rho}\,\hat{P}_a}{\text{Prob}(a)}}.$$

This is just the Born rule for mixed states.

<u>Sanity check</u>: $\sum_a \text{Prob}(a)$

$$= \sum_a \text{Tr}(\hat{\rho}\,\hat{P}_a)$$

$$= \text{Tr}\Big(\hat{\rho}\sum_a \hat{P}_a\Big) = \text{Tr}(\hat{\rho})$$

$$= 1$$

Ex $\hat{A} = \hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - 1 \qquad \lambda_+ = 1$$
$$\lambda_- = -1$$

Eigenvectors

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \begin{matrix} -1 - i^2 = 0 \\ i - i = 0 \end{matrix}$$

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \begin{matrix} +1 + i^2 = 0 \\ i - i = 0 \end{matrix}$$

$$\hat{P}_+ = |\psi_+\rangle\langle\psi_+|$$
$$= \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}(1, -i) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$\hat{P}_- = |\psi_-\rangle\langle\psi_-| = \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}(1, i) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Sanity check

$$\hat{P}_+^2 = \frac{1}{4}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2 & -2i \\ 2i & 2 \end{pmatrix} = P_+$$

$$\hat{P}_-^2 = \hat{P}_-$$

$$\hat{P}_+ + \hat{P}_- = \hat{I}$$

$$\hat{P}_+^\dagger = \hat{P}_+$$

$$\hat{P}_-^\dagger = \hat{P}_-$$

$$\hat{P}_+ \hat{P}_- = \frac{1}{4}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$\hat{\rho} = |+\rangle\langle+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\mathrm{Prob}(+) = \mathrm{Tr}\left(\hat{P}_+ \hat{\rho}\right)$$

$$= \mathrm{Tr}\left(\frac{1}{4}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{4}\mathrm{Tr}\begin{pmatrix} 1+i & 1+i \\ 1-i & 1-i \end{pmatrix}$$

$$= \frac{1}{2}$$

$$\mathrm{Prob}(-) = \mathrm{Tr}\left(\hat{P}_- \hat{\rho}\right) = \frac{1}{2}$$

$$\hat{\rho}_+ = \frac{\hat{P}_+ \hat{\rho} \hat{P}_+}{\mathrm{Prob}(+)} = 2\hat{P}_+ \hat{\rho} \hat{P}_+$$

$$= \frac{1}{4}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1+i & 1+i \\ 1-i & 1-i \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1+i-i+1 & i-1+1+i \\ 1-i-i-1 & i+1+1-i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$= \hat{P}_+ = |\psi_+\rangle\langle\psi_+|$$

Similarly $\hat{P}_- = |\psi_-\rangle\langle\psi_-|$

In this course we will need a <u>generalized</u> notion of <u>measurement</u> covers the case when the measurement operators are <u>not projectors</u>.

Positive operator-valued measures

(POVMs)

First a quick note on the unitary evolution of density matrices

Recall $|\psi\rangle \mapsto U|\psi\rangle$

For $\hat{\rho} \in D(\mathcal{H})$ $\quad \hat{\rho} \mapsto U \rho U^\dagger$

This makes sense, consider $\hat{\rho} = |\psi\rangle\langle\psi|$

$|\psi\rangle\langle\psi| \mapsto U|\psi\rangle\langle\psi|U^\dagger$

Easy way to remember where the dagger goes!

Consider the bipartite system

$$|\psi\rangle_{AB} = \left(\alpha\,|0\rangle_A + \beta\,|1\rangle_A\right) \otimes |1\rangle_B$$

Suppose that $|\psi\rangle_{AB}$ undergoes unitary evolution

$$\mathcal{U} : \alpha\,|0\rangle_A|0\rangle_B + \beta\,|1\rangle_A|0\rangle_B$$

$$\mapsto \alpha\,|0\rangle_A|0\rangle_B + \beta\,|1\rangle_A|1\rangle_B = |\psi'\rangle_{AB}$$

Suppose we measure the observable

$\hat{Z}_B = \hat{I}_A \otimes \hat{Z}_B$ Skip this in lecture

i.e. do a projective measurement of

$\hat{Z}$ on subsystem B.

$$\begin{array}{c} \hat{Z} \\ \parallel \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

Projectan (on B) are

$$\hat{P}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \hat{P}_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{I} \otimes \hat{P}_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \hat{P}_+ & 0 \\ 0 & \hat{P}_+ \end{pmatrix}$$

$$\hat{I} \otimes \hat{P}_- = \begin{pmatrix} \hat{P}_- & 0 \\ 0 & \hat{P}_- \end{pmatrix}$$

$$\text{Rrob}(+) = {}_{AB}\langle \psi' | \hat{I} \otimes \hat{P}_+ | \psi' \rangle_{AB}$$

$$= (\alpha^*, 0, 0, \beta^*) \; \hat{I} \otimes \hat{P}_+ \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{pmatrix}$$

$$= |\alpha|^2$$

$$\text{Rrob}(-) = |\beta|^2$$

$$|\psi_+\rangle_{AB} = \frac{\hat{P}_+ |\psi'\rangle_{AB}}{\sqrt{\text{Prob}(+)}}$$

$$= \frac{\alpha}{\sqrt{|\alpha|^2}} |0\rangle_A |0\rangle_B$$

$$|\psi_-\rangle_{AB} = \frac{\beta}{\sqrt{|\beta|^2}} |1\rangle_A |1\rangle_B$$

$\hat{I} \otimes \hat{P}_+$ acts "like a projector"

$$\hat{\rho}_+ = \frac{|\alpha|^2}{|\alpha|^2} |00\rangle\langle 00|_{AB}$$

$$\text{Tr}_B(\hat{\rho}_+) = |0\rangle\langle 0|_A$$

$$\text{Tr}_B(\hat{\rho}_-) = |1\rangle\langle 1|_A$$

Now suppose instead we measure

$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on qubit B.

$$\hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = |+\rangle\langle +|_B$$

$$\hat{P}_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = |-\rangle\langle -|_B$$

$$\text{Prob}(+) = {}_{AB}\langle \psi' | \hat{I}_A \otimes |+\rangle_B\langle +| \, |\psi'\rangle_{AB}$$

$$\left[\left(|0\rangle_A\langle 0| + |1\rangle_A\langle 1| \right) \otimes |+\rangle_B\langle +| \right] \alpha |0\rangle_A \otimes |0\rangle_B + \beta |1\rangle_A \otimes |1\rangle_B$$

$$\frac{\alpha}{\sqrt{2}} |0\rangle_A \otimes |+\rangle_B + \frac{\beta}{\sqrt{2}} |1\rangle_A \otimes |+\rangle_B$$

$$\text{Prob}(+) = \frac{|\alpha|^2}{2} + \frac{|\beta|^2}{2} = \frac{1}{2}$$

$$\langle +|0\rangle = \langle +|1\rangle = \frac{1}{\sqrt{2}}$$

$$\text{Prob}(-) = \frac{1}{2}$$

$$\langle -|0\rangle = \frac{1}{\sqrt{2}}$$

$$\langle -|1\rangle = -\frac{1}{\sqrt{2}}$$

$$\hat{I}_A \otimes |-\rangle_B\langle -| \, |\psi'\rangle_{AB}$$

$$= \frac{\alpha}{\sqrt{2}} |0\rangle_A \otimes |-\rangle_B - \frac{\beta}{\sqrt{2}} |1\rangle_A \otimes |-\rangle_B$$

$$|\psi_+\rangle_{AB} = \frac{\hat{I}_A \otimes |+\rangle_B\langle +| \, |\psi'\rangle_{AB}}{\sqrt{\text{Prob}(+)}}$$

$$= \alpha |0+\rangle_{AB} + \beta |1+\rangle_{AB}$$

$$|\psi_-\rangle_{AB} = \alpha|0-\rangle_{AB} - |1-\rangle_{AB}$$

$$\hat{\rho}_+ = |\psi_+\rangle\langle\psi_+|_{AB}$$

$$\mathrm{Tr}_B(\hat{\rho}_+) = \mathrm{Tr}_B\Big(|\alpha|^2 |0+\rangle\langle 0+|_{AB} + \alpha\beta^* |0+\rangle\langle 1+|_{AB}$$
$$+ \beta\alpha^* |1+\rangle\langle 0+|_{AB} + |\beta|^2 |1+\rangle\langle 1+|_{AB} \Big)$$

$$= \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

$$= |\phi\rangle\langle\phi|_A$$

whee $|\phi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$

$$\hat{\rho}_- = |\psi_-\rangle\langle\psi_-|_{AB}$$

$$\mathrm{Tr}_B(\hat{\rho}_-) = \mathrm{Tr}_B\Big(|\alpha|^2 |0-\rangle\langle 0-|_{AB} - \alpha\beta^* |0-\rangle\langle 1-|_{AB}$$
$$- \beta\alpha^* |1-\rangle\langle 0-|_{AB} + |\beta|^2 |1-\rangle\langle 1-|_{AB} \Big)$$

$$\mathrm{Tr}_B(\hat{\rho}_-) = |\chi\rangle\langle\chi|_A$$

where $|\chi\rangle_A = \alpha|0\rangle_A - \beta|1\rangle_A$

But $\langle\phi|\chi\rangle$

$$= (\alpha^*, \beta^*)\begin{pmatrix}\alpha \\ -\beta\end{pmatrix}$$

$$= |\alpha|^2 - |\beta|^2 \overset{!}{=} 0$$ unless $|\alpha| = |\beta|$

Let us rewrite

$$|\psi'\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|11\rangle_{AB}$$

$$= \frac{1}{\sqrt{2}} \underbrace{\hat{I}_A}_{\hat{M}_+} |\phi\rangle_A \otimes |+\rangle_B + \frac{1}{\sqrt{2}} \underbrace{\hat{Z}_A}_{\hat{M}_-} |\phi\rangle_A \otimes |-\rangle_B$$

only acts on A!

$$\frac{1}{\sqrt{2}}\left(\alpha|0+\rangle_{AB} + \beta|1+\rangle_{AB} + \alpha|0-\rangle_{AB} - \beta|1-\rangle_{AB}\right)$$

$$\alpha|0\rangle_A \underbrace{\left(\frac{|+\rangle_B + |-\rangle_B}{\sqrt{2}}\right)}_{|0\rangle_B} + \beta|1\rangle_A \underbrace{\left(\frac{|+\rangle_B - |-\rangle_B}{\sqrt{2}}\right)}_{|1\rangle_B}$$

We can now read off that a measurement of $\hat{I}_A \otimes \hat{X}_B$ prepares the states

$$\frac{\frac{1}{\sqrt{2}}|\phi\rangle_A}{\sqrt{\mathrm{Prob}(+)}} = |\phi\rangle_A$$

and $\hat{Z}|\phi\rangle_A = |\chi\rangle_A$

Now suppose that subsystem B is a qudit with orthonormal basis

$\{|a\rangle : a \in \{0, 1, ..., d-1\}\}$

Suppose that we start in the state $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |0\rangle_B$

$$\mathcal{U} : |\psi\rangle_{AB} \mapsto \sum_a \hat{M}_a |\phi\rangle_A \otimes |a\rangle_B$$

$$= |\psi'\rangle_{AB}$$

$${}_{AB}\langle\psi'|\psi'\rangle_{AB} = {}_{AB}\langle\psi|\psi\rangle_{AB} = 1$$

$$\sum_{a,b} {}_A\langle\phi| \hat{M}_a^\dagger \hat{M}_b |\phi\rangle_A \, {}_B\langle a|b\rangle_B = 1$$

$$= \sum_a {}_A\langle\phi| \hat{M}_a^\dagger \hat{M}_a |\phi\rangle_A = 1$$

$|\phi\rangle$ is arbitrary $\Rightarrow \sum_a \hat{M}_a^\dagger \hat{M}_a = \hat{I}$

Born rule for $\hat{P}_a = \hat{I}_A \otimes |a\rangle\langle a|_B$

$$\text{Prob}(a) = \| \hat{M}_a |\phi\rangle \|^2 = \langle \phi | \hat{M}_a^\dagger \hat{M}_a | \phi \rangle$$

$$= \| I_A \otimes |a\rangle\langle a|_B \; |\psi'\rangle_{AB} \|^2$$

$$= \| \hat{M}_a |\phi\rangle_A \otimes |a\rangle_B \|^2$$

$$= \| \hat{M}_a |\phi\rangle_A \|^2 \; \| |a\rangle_B \|^2$$

$$|\phi_a\rangle = \frac{\hat{M}_a |\phi_a\rangle}{\sqrt{\text{Prob}(a)}}$$

$$\text{Prob}(b|a) = \| \hat{M}_b |\hat{\phi}_a\rangle \|^2$$

$$= \frac{\| \hat{M}_b \hat{M}_a |\phi\rangle \|^2}{\text{Prob}(a)}$$

$$= \frac{\| \hat{M}_b \hat{M}_a |\phi\rangle \|^2}{\| \hat{M}_a |\phi\rangle \|^2}$$

If $\hat{M}_b \hat{M}_a = \delta_{ab} \hat{M}_a$ then we recover our orthogonal projectors

If instead the initial state on A is a density matrix $\hat{\rho}$

$$\text{Prob}(a) = \text{Tr}(\hat{\rho}\,\hat{M}_a^\dagger \hat{M}_a)$$

$$= \text{Tr}(\hat{\rho}\,\hat{E}_a)$$

What do we know about the $\hat{E}_a$?

$$\sum_a \hat{E}_a = \sum_a \hat{M}_a^\dagger \hat{M}_a = \hat{I}$$ Complete

$$\hat{E}_a^\dagger = (\hat{M}_a^\dagger \hat{M}_a)^\dagger = (\hat{M}_a^\dagger \hat{M}_a) = \hat{E}_a$$

Hermitian

$$\langle \phi | \hat{E}_a | \phi \rangle$$

$$= \| \hat{M}_a |\phi\rangle \|^2 \geq 0$$ Positive

Any set of operators w/ these properties is called a positive operator - valued measure (POVM).

Going back to our example

$$M_+ = \frac{1}{\sqrt{2}} \hat{I} \qquad M_- = \frac{1}{\sqrt{2}} \hat{Z}$$

$$\hat{E}_+ = \frac{1}{2} \hat{I} \qquad \hat{E}_- = \frac{1}{2} \hat{Z}^+ \hat{Z} = \frac{1}{2} I$$

Hermitian ✓

Complete ✓

Positive ✓

Lemma (Neumark)

Purification of a POVM!

Any POVM $\{\hat{E}_a\}_{a=0,\dots,d-1}$ can be realised by coupling the system A to an auxiliary qudit B and performing a projective measurement on system B.

Proof

Write $\hat{M}_a = \hat{U}_a \sqrt{\hat{E}_a}$

$\hat{E}_a$ is non negative and Hermitian

$\Rightarrow$ it has a non negative square root $\sqrt{\hat{E}_a}$

We have $M_a^\dagger M_a = \sqrt{\hat{E}_a}^\dagger \hat{U}_a^\dagger \hat{U}_a \sqrt{\hat{E}_a}$

$= \sqrt{\hat{E}_a}\sqrt{\hat{E}_a} = \hat{E}_a$

real ↙ ↙ orthonormal

Aside $\hat{E}_a = \sum_i \lambda_i |v_i\rangle\langle v_i|$

$\sqrt{\hat{E}_a} = \sum_i \sqrt{\lambda_i}\, |v_i\rangle\langle v_i|$

$(\sqrt{\hat{E}_a})^\dagger = \sum_i \underbrace{(\sqrt{\lambda_i})^\dagger}_{= \sqrt{\lambda_i}} |v_i\rangle\langle v_i|$

$\langle\psi| \sqrt{\hat{E}_a} |\psi\rangle = \sum_i \sqrt{\lambda_i}\, |\langle v_i|\psi\rangle|^2 \geq 0$

$U: \rho_A \otimes |0\rangle\langle 0|_B$

$\mapsto \sum_a M_a \hat{\rho}_A M_a^\dagger \otimes |a\rangle\langle a|_B$

On pure states

$$U : |\psi\rangle_A \otimes |0\rangle_B$$

$$\mapsto \sum_a \hat{M}_a |\psi\rangle_A \otimes |a\rangle_B$$

$$_B\langle 0| \otimes {}_A\langle\psi| \; U^\dagger U \; |\psi\rangle_A \otimes |0\rangle_B$$

$$= \sum_{a,b} {}_B\langle a| \otimes {}_A\langle\psi| \hat{M}_a^\dagger \hat{M}_b |\psi\rangle_B \otimes |b\rangle_B$$

$$= \sum_a {}_A\langle\psi| \hat{M}_a^\dagger M_a |\psi\rangle_A$$

$$= \langle\psi| \underbrace{\sum_a \hat{M}_a^\dagger \hat{M}_a}_{\sum_a \hat{E}_a = \hat{I}} |\psi\rangle_A = {}_A\langle\psi|\psi\rangle_A$$

Preserves inner products

Technically an isometry but can be extended to a unitary on the whole space.

Projective measurement on B

$$\hat{I}_A \otimes |a\rangle\langle a|_B$$

$$\mathrm{Prob}(a) = \mathrm{Tr}\Big(\hat{I}_A \otimes |a\rangle\langle a|_B$$

$$\sum_b \hat{M}_b \hat{\rho}_A \hat{M}_b^\dagger \otimes |b\rangle\langle b|_B \Big)$$

$$= \mathrm{Tr}\big(\hat{M}_a \hat{\rho}_A \hat{M}_a^\dagger \otimes |a\rangle\langle a|_B \big)$$

$$= \mathrm{Tr}\big(\hat{M}_a \hat{\rho}_A \hat{M}_a^\dagger \big) \underbrace{\mathrm{Tr}\big(|a\rangle\langle a|_B \big)}_{1}$$

$$= \mathrm{Tr}\big(\hat{\rho}_A \hat{M}_a^\dagger \hat{M}_a \big)$$

$$= \mathrm{Tr}\big(\hat{\rho}_A \hat{E}_a \big)$$

□

What is the post-measurement state?

$$\rho'_A = \mathrm{Tr}_B\Big(\hat{I}_A \otimes |a\rangle_B\langle a|$$

$$\sum_b \hat{M}_b \hat{\rho}_A \hat{M}_b^{\dagger}\ |b\rangle_B\langle b|$$

$$\hat{I}_A \otimes |a\rangle_B\langle a| \Big) \Big/ Prob(a)$$

$$= \mathrm{Tr}_B\Big(\hat{M}_a \hat{\rho}_A \hat{M}_a^{\dagger} \otimes |a\rangle_B\langle a| \Big) / Prob(a)$$

$$= \sum_b \hat{M}_a \hat{\rho}_A \hat{M}_a^{\dagger}\ {}_B\langle b|a\rangle_B\langle a|b\rangle_B \ / Prob(a)$$

$$= \frac{M_a \hat{\rho}_A M_a^{\dagger}}{Prob(a)}$$

$$= \frac{\hat{U}\sqrt{\hat{E}_a}\ \hat{\rho}_A\left(\hat{U}\sqrt{\hat{E}_a}\right)^{\dagger}}{Prob(a)}$$

$$= \hat{U}\left[\frac{\sqrt{\hat{E}_a}\ \hat{\rho}_A\sqrt{\hat{E}_a}}{Prob(c)}\right]\hat{U}^{\dagger}$$

$\hat{U}$ is arbitrary so the post-meas. state is not uniquely determined! It depends on how one implements the POVM.

Ex

$$|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$$

$$\alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B$$

$$= \underbrace{P_{\underset{A}{|0\rangle\langle 0|}}}_{M_0} |\psi\rangle_A |0\rangle_B + \underbrace{P_{\underset{A}{|1\rangle\langle 1|}}}_{M_1} |\psi\rangle_A |1\rangle_B$$

POVM w/ elements $M_0 M_0^\dagger = M_0$

and $M_1 M_1^\dagger = M_1$

The measurement problem

Let us return to the axioms of Q.M.

1. States are density matrices

$$\hat{\rho} \in D(\mathcal{H})$$

$$\hat{\rho}^\dagger = \hat{\rho}\,, \quad \hat{\rho} \geq 0\,, \quad \mathrm{Tr}(\hat{\rho}) = 1$$

2. Composite systems are formed from tensor products of their component systems

3. Time evolution is unitary $\hat{U} \in B(\mathcal{H})$

$$\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^\dagger \qquad \hat{U}\hat{U}^\dagger = \hat{I}$$

4. Measurement via POVMs

$$\{\hat{M}_a\} \quad \sum_a \hat{M}_a^\dagger \hat{M}_a = \hat{I} \qquad \hat{M}_a \in B(\mathcal{H})$$

$$p(a) = \mathrm{Tr}(\hat{M}_a^\dagger \hat{M}_a \hat{\rho})$$

Post-measurement state

$$\hat{\rho}' = \frac{M_a \hat{\rho} M_a^{+}}{Tr(\hat{M}_a^{+}\hat{M}_a \hat{\rho})}$$

Measurement problem:

how to get 4. from 1., 2., 3. ?

1. Many-worlds

There is no 4!

2. Psi-epistemic

$\hat{\rho}$ represents a state of knowledge

Other options: decoherence?

pilot wave (Bohm)?
de Broglie