Lecture I : Quantum States

Density matrices (operators)

Useful to describe

- Probabilistic preparations of quantum states
- Subsystems of quantum states

We will need both in this course

Def Hilbert space $\mathcal{H}$

Complex vector space with an inner product

$\langle \cdot | \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$

such that

1. Linear $|\psi\rangle, |\phi\rangle, |\chi\rangle \in \mathcal{H}$ $\alpha, \beta \in \mathbb{C}$

$$(\alpha\langle\psi| + \beta\langle\phi|)(|\chi\rangle)$$ (1st argument)

$$= \alpha\langle\psi|\chi\rangle + \beta\langle\phi|\chi\rangle$$

2. Conjugate symmetry

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$$

3. Positive definite

$\langle\psi|\psi\rangle \geq 0$ and equal to 0

iff $|\psi\rangle = 0$

Norm

$$\| |\psi\rangle \| = \sqrt{\langle\psi|\psi\rangle}$$

4. Completeness (inf dim spaces)

Any Cauchy sequence of states in $\mathcal{H}$ converges to another state in $\mathcal{H}$.

$\{|\psi_i\rangle\}$ s.t. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t.

$$\forall i, j > N \quad \| |\psi_i\rangle - |\psi_j\rangle \| < \varepsilon$$

Ex

$\mathcal{H} = \mathbb{C}^2$ Qubit $|0\rangle, |1\rangle$

$\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ n qubits

You will see examples of inf dim Hilbert spaces in the Q.C. course.

Def Bounded linear operators on $\mathcal{H}$, $B(\mathcal{H})$

Operator $\hat{\sigma} : \mathcal{H} \to \mathcal{H}$ in $B(\mathcal{H})$ if

1. Linear

$$\hat{\sigma}\left(\alpha|\psi\rangle + \beta|\phi\rangle\right)$$

$$= \alpha\hat{\sigma}|\psi\rangle + \beta\hat{\sigma}|\phi\rangle$$

2. Bounded

$\exists\, C \in \mathbb{R}^+$ s.t.

$$\| \hat{\sigma} |\psi\rangle \| \leq C \| |\psi\rangle \| \quad \forall\, |\psi\rangle \in \mathcal{H}$$

$$\| \hat{\sigma} |\psi\rangle \| = \sqrt{\langle\psi|\hat{\sigma}^{+}\hat{\sigma}|\psi\rangle}$$

<u>Ex</u>

2x2 complex matrices

$$\mathcal{H} = \mathbb{C}^2$$

$$B(\mathcal{H}) = M_2(\mathbb{C})$$

$$\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$$

$$B(\mathcal{H}) = M_{2^n}(\mathbb{C})$$

<u>Def</u> Density matrices (operators)

$\hat{\rho} \in B(\mathcal{H})$ is a density matrix (in $D(\mathcal{H})$) if

1. $\hat{\rho}^{+} = \hat{\rho}$ Hermitian

2. $\mathrm{Tr}(\hat{\rho}) = 1$ Normalised

3. $\hat{\rho} \geq 0$ Positive

$\forall |\psi\rangle \in \mathcal{H} \quad \langle \psi | \rho | \psi \rangle \geq 0$

Ex

$$\mathcal{H} = \mathbb{C}^2$$

$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\hat{\rho}^{+} = \hat{\rho}$ ✓

$\mathrm{Tr}(\hat{\rho}) = \frac{1}{2} + \frac{1}{2} = 1$ ✓

$\hat{\rho} \geq 0 \quad \forall |\psi\rangle \in \mathcal{H} \quad \langle \psi | \hat{\rho} | \psi \rangle$ ✓

$$= \langle \psi | \psi \rangle = \| |\psi\rangle \|^2 \geq 0$$

Lemma All $\hat{\rho} \in D(\mathcal{H})$ have real eigenvalues that sum to 1.

Proof

$\in \mathbb{C}$

Suppose $\hat{\rho} |\phi\rangle = \lambda |\phi\rangle$

$$\langle \phi | \hat{\rho} | \phi \rangle = \lambda \langle \phi | \phi \rangle$$

$$\overset{\text{Hermitian}}{\parallel} \qquad\qquad \parallel$$

$$\langle \phi | \hat{\rho}^{+} | \phi \rangle = \lambda^{*} \langle \phi | \phi \rangle$$

$$\Rightarrow \lambda \in \mathbb{R} \quad \left(\langle \phi | \phi \rangle \geq 0 \right)$$

$$\langle \phi | \rho | \phi \rangle \geq 0$$

$$\lambda \langle \phi | \phi \rangle \geq 0$$

$$\langle \phi | \phi \rangle \geq 0 \Rightarrow \lambda \geq 0$$

$$\mathrm{Tr}(\hat{\rho}) = 1$$

$$\mathrm{Tr}(U D U^\dagger) = 1$$

$$\ulcorner D = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$$

U diagonalizes $\hat{\rho}$ $\lrcorner$

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$$

$$\Rightarrow \mathrm{Tr}(U D U^\dagger) = \mathrm{Tr}(\underbrace{U^\dagger U}_{I} D)$$

$$= \mathrm{Tr}(D)$$

$$= \sum_i \lambda_i = 1$$

$\square$

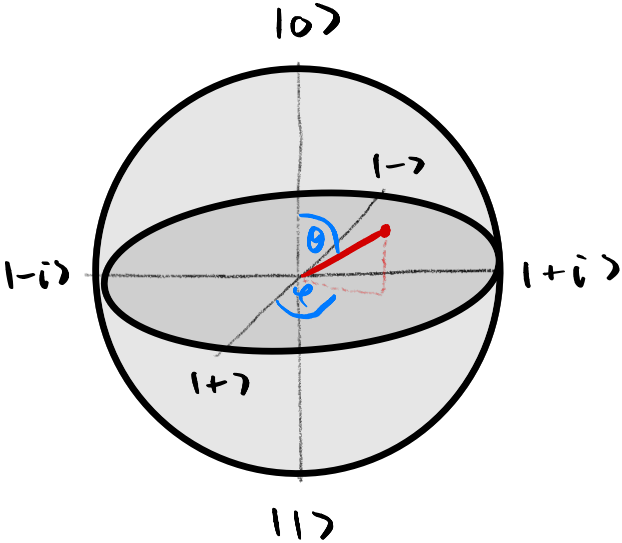
Def $\hat{\rho} \in D(\mathcal{H})$ is pure if

$\hat{\rho} = |\psi\rangle\langle\psi|$ projector onto $|\psi\rangle$

$$\Rightarrow \hat{\rho}^2 = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi| = \hat{\rho}$$

Bloch sphere

You saw the Bloch sphere last lecture



Lemma Any 1 qubit density operator can be written as

$$\hat{\rho}(\mathbf{r}) = \frac{1}{2}\left(\hat{I} + \hat{r} \cdot \hat{\mathbf{P}}\right)$$

where $\mathbf{r} \in \mathbb{R}^3$ $|\mathbf{r}| \leq 1$

$$\hat{\mathbf{P}} = (\hat{X}, \hat{Y}, \hat{Z})$$

Proof

$\hat{\rho}^{+} = \hat{\rho}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$

$a, b, c, d \in \mathbb{C}$ $= \begin{pmatrix} a & x+iy \\ x-iy & d \end{pmatrix}$

Above places 4 constraints

$\Rightarrow$ 4 real number degrees of freedom

We can expend $\hat{\rho}$ in the basis

$\{\hat{X}, \hat{Y}, \hat{Z}, \hat{I}\}$

$$\hat{\rho} = \frac{1}{2}\left(r_0 \hat{I} + r_1 \hat{X} + r_2 \hat{Y} + r_3 \hat{Z}\right)$$

$$\mathrm{Tr}(\hat{\rho}) = 1$$

$$\mathrm{Tr}(\hat{X}) = \mathrm{Tr}(\hat{Y}) = \mathrm{Tr}(\hat{Z}) = 0$$

$$\Rightarrow r_0 = 1$$

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + r_1 \hat{X} + r_2 \hat{Y} + r_3 \hat{Z}\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1 + r_3 & r_1 - i r_2 \\ r_1 + i r_2 & 1 - r_3 \end{pmatrix}$$

$\mathrm{Tr}(\hat{\rho}) = 1 \Rightarrow$ eigenvalues of $\hat{\rho}$ cannot both be negative

If $\mathrm{Det}(\hat{\rho}) = \frac{1}{4}\left(1 - P_1^2 - P_2^2 - P_3^2\right) \geqslant 0$

then both eigenvalues are positive.

$\Rightarrow \quad r_1^2 + r_2^2 + r_3^2 \leqslant 1$

$$\hat{\rho}(P_1, P_2, P_3) = \frac{1}{2}\left(\hat{I} + r_1 \hat{X} + r_2 \hat{Y} + r_3 \hat{Z}\right)$$

$$\hat{\rho}(\underline{\hat{P}}) = \frac{1}{2}\left(\hat{I} + \underline{r} \cdot \underline{\hat{P}}\right) \qquad |\underline{r}| \leqslant 1$$

$\square$

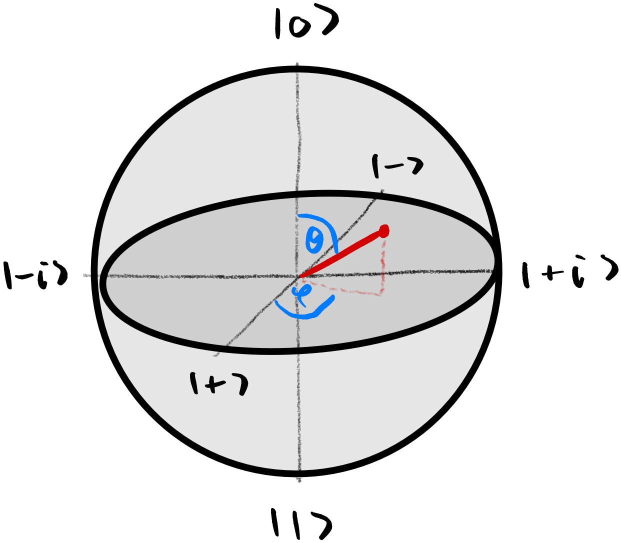
Corollary

There is an isomorphism between 1q density matrices and points in the unit 3-ball

$\{ \underline{r} \in \mathbb{R}^3 : |\underline{r}| \leqslant 1 \}$

called the Bloch "sphere"

Bloch vector



$|\underline{r}| = 1$ pure state e.g. $|0\rangle\langle 0|$

$|\underline{r}| < 1$ mixed state e.g. $\frac{1}{2}\hat{I}$

Schmidt decomposition

e.g. $\mathbb{C}^2 \otimes \mathbb{C}^2$

Lemma Suppose $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Then we can write any

$|\psi\rangle_{AB} \in \mathcal{H}$ in the form

$$|\psi\rangle_{AB} = \sum_{i=1}^{r} \lambda_i \, |\varphi_i\rangle \otimes |\chi_i\rangle$$

where $\{|\varphi_i\rangle\}$ $\{|\chi_i\rangle\}$ are orthonormal bases for $\mathcal{H}_A$ & $\mathcal{H}_B$, respectively,

$\lambda_i \in \mathbb{R}^+$, $r \leq \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$

Schmidt rank

Proof

matrix

$r \leq \text{rank}(\psi)$ rows

$= \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$

cols

$|\psi\rangle_{AB} \in \mathcal{H}$

$$= \sum_{i,\mu}^{r} \psi_{i\mu} |i\rangle_A \otimes |\mu\rangle_B$$

$\{|i\rangle_A\}$ orthonormal basis for $\mathcal{H}_A$

${}_A\langle i | j \rangle_A = \delta_{ij}$ $\qquad \{|\mu\rangle_B\}$ for $\mathcal{H}_B$

Change of basis $|\tilde{x}_i\rangle := \sum_\mu \psi_{i\mu} |\mu\rangle_B$

Not orthonormal in general

$$|\psi\rangle_{AB} = \sum_i |i\rangle_A \otimes |\tilde{x}_i\rangle_B$$

Suppose $\hat{\rho}_A = \text{Tr}_B (|\psi\rangle\langle\psi|_{AB})$

$$= \sum_i p_i |i\rangle\langle i|_A$$

By direct calculation

$$\hat{\rho}_A = \mathrm{Tr}_B\left(|\psi\rangle_{AB}\right)$$

$$= \mathrm{Tr}_B\left(\sum_{i,j} |i\rangle_A\langle j| \otimes |\tilde{x}_i\rangle_B\langle \tilde{x}_j|\right)$$

$$= \sum_{i,j} {}_B\langle \tilde{x}_j|\tilde{x}_i\rangle_B \; |i\rangle_A\langle j|$$

NB $$\mathrm{Tr}\left(|\tilde{x}_i\rangle_B\langle \tilde{x}_j|\right)$$

orthonormal basis

$$= \sum_k {}_B\langle k|\tilde{x}_i\rangle\langle \tilde{x}_j|k\rangle_B$$

$$= \sum_k {}_B\langle \tilde{x}_j|k\rangle_B \; {}_B\langle k|\tilde{x}_i\rangle_B$$

$$= {}_B\langle \tilde{x}_j| \underbrace{\sum_k |k\rangle_B\langle k|}_{\hat{I}} |\tilde{x}_i\rangle_B$$

$$= {}_B\langle \tilde{x}_j|\tilde{x}_i\rangle_B$$

$$\sum_i p_i |i\rangle\langle i|_A = \sum_{i,j} {}_B\langle \tilde{x}_j | \tilde{x}_i \rangle_B \, |i\rangle\langle j|_A$$

$$\Rightarrow \quad {}_B\langle \tilde{x}_j | \tilde{x}_i \rangle_B = p_i \, \delta_{ij}$$

$\{ |\tilde{x}_i\rangle_B \}$ are orthogonal in fact

Define $\quad |x_i\rangle_B := p_i^{-1/2} \, |\tilde{x}_i\rangle_B$

$${}_B\langle x_i | x_j \rangle_B = \frac{1}{p_i} \, p_i \, \delta_{ij} = \delta_{ij}$$ orthonormal

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i} \, |i\rangle_A \otimes |x_i\rangle_B$$

is the Schmidt decomposition of $|\psi\rangle_{AB}$

$\sqrt{p_i}$ are the singular values of

$|\psi\rangle_{AB}$

Ex $|\psi\rangle_{AB} = \frac{1}{2}\left(|00\rangle_{AB} + |01\rangle_{AB} + |10\rangle_{AB} - |11\rangle_{AB}\right)$

$$\hat{\rho}_A = \mathrm{Tr}_B\left(|\psi\rangle\langle\psi|_{AB}\right)$$

$$= {}_B\langle 0|\psi\rangle\langle\psi|_{AB}|0\rangle_B + {}_B\langle 1|\psi\rangle\langle\psi|_{AB}|1\rangle_B$$

$$= |0\rangle\langle 0|_A + \cancel{|0\rangle\langle 1|_A} + \cancel{|1\rangle\langle 0|_A} + |1\rangle\langle 1|_A + |0\rangle\langle 0|_A - \cancel{|0\rangle\langle 1|_A} - \cancel{|1\rangle\langle 0|_A} + |1\rangle\langle 1|_A$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sum_i \frac{1}{2}|i\rangle\langle i|$$

$$p_0 = p_1 = \frac{1}{2}$$

$$|\psi\rangle_{AB} = \frac{1}{2}\left(|00\rangle_{AB} + |01\rangle_{AB} + |10\rangle_{AB} - |11\rangle_{AB}\right)$$

$$|\tilde{0}\rangle_B = \sum_\mu \psi_{0\mu} |\mu\rangle_B$$

$$= \frac{1}{2}\left(|0\rangle_B + |1\rangle_B\right)$$

$$\frac{1}{\sqrt{p_0}} = \sqrt{2}$$

$$|\tilde{1}\rangle_B = \frac{1}{2}\left(|0\rangle_B - |1\rangle_B\right)$$

$$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$|0'\rangle_B = \frac{1}{\sqrt{p_0}}|\tilde{0}\rangle_B = \frac{1}{\sqrt{2}}\left(|0\rangle_B + |1\rangle_B\right) = |+\rangle_B$$

$$|1'\rangle_B = |-\rangle_B$$

$$\Rightarrow \quad |\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |+\rangle_B + |1\rangle_A \otimes |-\rangle_B\right)$$

Convexity of mixed states

Lemma

Suppose $\hat{\rho}_1$ & $\hat{\rho}_2 \in D(\mathcal{H})$

$$x \in [0,1]$$

then

$$\hat{\rho}(x) = x\hat{\rho}_1 + (1-x)\hat{\rho}_2 \in D(\mathcal{H})$$

Proof

$$\hat{\rho}(x)^\dagger = x\hat{\rho}_1^\dagger + (1-x)\hat{\rho}_2^\dagger = \hat{\rho}(x)$$

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}(x)) &= \mathrm{Tr}\left(x\hat{\rho}_1 + (1-x)\hat{\rho}_2\right) \\
&= \mathrm{Tr}(x\hat{\rho}_1) + \mathrm{Tr}((1-x)\hat{\rho}_2) \\
&= x\,\mathrm{Tr}(\hat{\rho}_1) + (1-x)\,\mathrm{Tr}(\hat{\rho}_2) \\
&= x + (1-x) = 1
\end{aligned}$$

$\langle \psi | \hat{\rho} | \psi \rangle$

$$= x \underbrace{\langle \psi | \hat{\rho}_1 | \psi \rangle}_{\geq 0} + (1-x) \underbrace{\langle \psi | \hat{\rho}_2 | \psi \rangle}_{\geq 0}$$

≥ 0

$\square$

<u>Corollary</u> $D(\mathcal{H})$ is a convex set.

For a pure state $|\phi\rangle$ $x = 1$

$\hat{\rho} = |\phi\rangle\langle\phi|$

Pure states are <u>extremal points</u> of the set.

We can always write a mixed state in the basis in which it is diagonal.

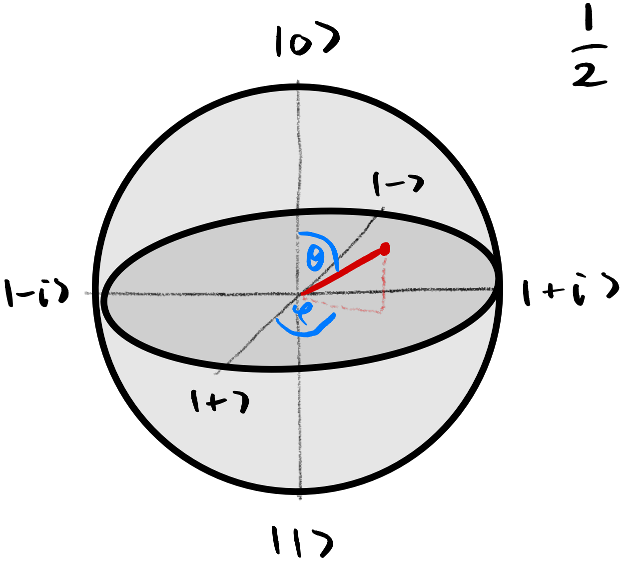
$$\hat{\rho} = \sum_i p_i |i\rangle\langle i|$$

This gives us a way to prepare $\hat{\rho}$. Imagine a machine that prepares $|i\rangle$ w/ probability i etc.

This prepares the state $\hat{\rho}$.

NB The decomposition is not unique

$$\frac{1}{2}\hat{I} = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$



But $\hat{I} = \frac{1}{2}\left(|+\rangle\langle+| + |-\rangle\langle-| \right)$

Check: $|+\rangle\langle+| = \frac{1}{2}\Big(|0\rangle\langle 0| + |0\rangle\langle 1|$

$+ |1\rangle\langle 0| + |1\rangle\langle 1|\Big)$

$|-\rangle\langle-| = \frac{1}{2}\Big(|0\rangle\langle 0| - |0\rangle\langle 1|$

$- |1\rangle\langle 0| + |1\rangle\langle 1|\Big)$

You can check it for $|+i\rangle$ & $|-i\rangle$!

In general, we can write the non-extremal points in infinitely many ways! $(x \in [0,1])$

Purification

We have seen that we can always write a mixed state as

$$\hat{\rho}_A = \sum_i p_i \, |\varphi_i\rangle\langle\varphi_i|_A \qquad \sum_i p_i = 1$$

(diagonalize the matrix)

We can construct the purification of $\hat{\rho}$ as follows

$$|\phi\rangle_{AB} = \sum_i \sqrt{p_i} \, |\varphi_i\rangle_A \otimes |x_i\rangle_B$$

where $\langle x_i | x_j \rangle = \delta_{ij}$ Schmidt decomposition

Let's check!

$$\mathrm{Tr}_B\left(|\phi\rangle\langle\phi|_{AB} \right)$$

$$= \sum_k {}_B\langle x_k | \phi\rangle\langle\phi|_{AB} | x_k\rangle_B$$

$$= \sum_{i,j,k} \sqrt{p_i}\sqrt{p_j}\ {}_B\langle x_k | \left(|x_i\rangle_B \otimes |\varphi_i\rangle_A \right) \left({}_A\langle \varphi_j | \otimes {}_B\langle x_j | \right) | x_k\rangle_B$$

$$= \sum_{i,j,k} \sqrt{p_i}\sqrt{p_j}\,\delta_{ki}\,\delta_{kj}\,|\varphi_i\rangle\langle\varphi_j|_A$$

$$= \sum_k p_i |\varphi_i\rangle\langle\varphi_i|_A$$

Hughston, Jozsa, Wootters

Lemma Any two purifications are related by a unitary acting on the purifying system.

Let $\hat{\rho}_A = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|_A$

$$|\phi_1\rangle_{AB} = \sum_i \sqrt{p_i}\, |\varphi_i\rangle_A \otimes |\chi_i\rangle_B$$

$$\langle \chi_i | \chi_j \rangle = \delta_{ij}$$

But we can also prepare $\hat{\rho}_A$ in a different way

$$|\phi_2\rangle_{AB} = \sum \sqrt{p_i}\, |\varphi_i\rangle_A \otimes |\tau_i\rangle_B$$

Change of basis $\hat{u}|\chi_i\rangle = |\tau_i\rangle$

$$\langle \tau_i | \tau_j \rangle = \langle \chi_i | \hat{u}^\dagger \hat{u} | \chi_j \rangle = \delta_{ij}$$

$$\Rightarrow \quad |\phi_1\rangle_{AB} = (\hat{I} \otimes \hat{u})\, |\phi_2\rangle_{AB}$$

$\square$

Distinguishing quantum states

Def : Fidelity between $\hat{\rho}, \hat{\sigma} \in D(\mathcal{H})$

$$F(\hat{\rho}, \hat{\sigma}) = \mathrm{Tr}\left(\sqrt{\hat{\rho}^{1/2}\,\hat{\sigma}\,\hat{\rho}^{1/2}}\right)^2$$

Do the next lemma first!

Lemma : For $\hat{\rho} = |\psi\rangle\langle\psi|$, $\hat{\sigma} = |\phi\rangle\langle\phi|$

$$F(\hat{\rho}, \hat{\sigma}) = |\langle\psi|\phi\rangle|^2$$

Proof

$$\mathrm{Tr}\left(\sqrt{\hat{\rho}^{1/2}\,\hat{\sigma}\,\hat{\rho}^{1/2}}\right)^2$$

$$= \mathrm{Tr}\left(\sqrt{\hat{\rho}\,\hat{\sigma}\,\hat{\rho}}\right)^2 \qquad \hat{\rho}^2 = \hat{\rho}$$

$$= \mathrm{Tr}\left(\sqrt{|\psi\rangle\langle\psi|\phi\rangle\langle\phi|\psi\rangle\langle\psi|}\right)^2$$

$$\mathrm{Tr}\left(\sqrt{ |\psi\rangle\langle\psi|\phi\rangle \underbrace{\langle\phi|\psi\rangle}_{\langle\psi|\phi\rangle^*} \langle\psi| } \right)^2$$

$$= \mathrm{Tr}\left(\sqrt{ |\langle\psi|\phi\rangle|^2 \, |\psi\rangle\langle\psi| } \right)^2$$

$$= \mathrm{Tr}\left(|\langle\psi|\phi\rangle| \; |\psi\rangle\langle\psi| \right)^2$$

$\mathrm{Tr}(\lambda A) = \lambda \mathrm{Tr}(A)$

$$= |\langle\psi|\phi\rangle|^2 \; \underbrace{\mathrm{Tr}\left(|\psi\rangle\langle\psi| \right)^2}_{1}$$

$$= |\langle\psi|\phi\rangle|^2 \qquad \square$$

Lemma For $\hat{\rho} = |\psi\rangle\langle\psi|$ $\hat{\sigma} \in D(\mathcal{H})$

$$F(\hat{\rho}, \hat{\sigma}) = \langle\psi|\hat{\sigma}|\psi\rangle$$

Proof

$$\mathrm{Tr}\left(\sqrt{\hat\rho^{1/2}\,\hat\sigma\,\hat\rho^{1/2}}\right)^2$$

$$= \mathrm{Tr}\left(\sqrt{\hat\rho\,\hat\sigma\,\hat\rho}\right)^2 \qquad \hat\rho^2 = \hat\rho$$

$$= \mathrm{Tr}\left(\sqrt{|\psi\rangle\underbrace{\langle\psi|\,\hat\sigma\,|\psi\rangle}_{\text{scaler}}\langle\psi|}\right)^2$$

$$= \mathrm{Tr}\left(\sqrt{\langle\psi|\hat\sigma|\psi\rangle}\sqrt{|\psi\rangle\langle\psi|}\right)^2 \qquad |\psi\rangle\langle\psi| = (|\psi\rangle\langle\psi|)^2$$

$$= \langle\psi|\hat\sigma|\psi\rangle\,\mathrm{Tr}\left(|\psi\rangle\langle\psi|\right)^2$$

$$= \langle\psi|\hat\sigma|\psi\rangle \qquad \square$$

Thm (Uhlmann)

$$F(\hat{\rho}, \hat{\sigma}) = \max_{|\psi\rangle, |\phi\rangle} |\langle\psi|\phi\rangle|^2$$

where $|\psi\rangle$ and $|\phi\rangle$ are purifications of $\hat{\rho}$ and $\hat{\sigma}$, respectively.

Proof : Preskill Ch 2 2.6.1

Corollary

$$F(\hat{\rho}, \hat{\sigma}) \in [0, 1]$$

Cauchy Schwarz

$$|\langle\psi|\phi\rangle|^2 \leq \langle\psi|\psi\rangle\langle\phi|\phi\rangle$$

$$\leq 1$$

$$|\langle\psi|\phi\rangle|^2 \geq 0$$

But fidelity is not ideal as

$1-F$ is not a metric

Def metric $d : D(\mathcal{H}) \times D(\mathcal{H})$

$\rightarrow [0, \infty)$

1. $d(\hat{\rho}, \hat{\sigma}) \geq 0 \quad \forall \hat{\rho}, \hat{\sigma} \in D(\mathcal{H})$

$1 - F(\hat{\rho}, \hat{\sigma}) \in [0, 1]$ ✓

2. $d(\hat{\rho}, \hat{\sigma}) = 0$ iff $\hat{\rho} = \hat{\sigma}$

$F(\hat{\rho}, \hat{\sigma}) = 1$ iff $\hat{\rho} = \hat{\sigma}$

Uhlmann

as $|\langle \psi | \phi \rangle|^2 = 1$ iff $|\psi\rangle = e^{i\theta} |\phi\rangle$

$\Rightarrow \mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \mathrm{Tr}_B(|\phi\rangle\langle\phi|)$

✓

3. $d(\hat{\rho}, \hat{\sigma}) = d(\hat{\sigma}, \hat{\rho})$ ✓

$F(\hat{\rho}, \hat{\sigma}) = F(\hat{\sigma}, \hat{\rho})$ homework

4. $d(\hat{\rho}, \hat{\tau}) \leq d(\hat{\rho}, \hat{\sigma}) + d(\hat{\sigma}, \hat{\tau})$

triangle inequality ✗

homework

An example of a metric on $D(\mathcal{H})$ is the trace distance

Def: Trace distance between $\hat{\rho}, \hat{\sigma} \in D(\mathcal{H})$

$$D_{Tr}(\hat{\rho}, \hat{\sigma}) := \frac{1}{2} \mathrm{Tr} \| \hat{\rho} - \hat{\sigma} \|$$
$$= \frac{1}{2} \mathrm{Tr} \sqrt{(\hat{\rho} - \hat{\sigma})^{+}(\hat{\rho} - \hat{\sigma})}$$

You will show that $D_{Tr}(\hat{\rho}, \hat{\sigma})$ is a metric in the homework.

The fidelity and trace distance are related as follows

$$1 - \sqrt{F(\hat{\rho}, \hat{\sigma})} \leq D_{Tr}(\hat{\rho}, \hat{\sigma}) \leq \sqrt{1 - F(\hat{\rho}, \hat{\sigma})}$$